Pricing Defaultable Bonds in a Markov Modulated Market

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Basic Assumptions

- Company has a simple capital structure consisting of one debt obligation (or defaultable bond) and one type of equity.

- Debt and equity may be viewed as contingent claims on the firm’s assets.
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- Firm defaults if its asset value reaches a certain lower threshold.
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Debt and equity may be viewed as contingent claims on the firm’s assets.

Firm defaults if its asset value reaches a certain lower threshold.

Firm’s assets, default free zero coupon bonds and the defaultable claims are traded securities.
Market economy in a finite number of states, $\mathcal{X} = \{1, 2, \ldots, k\}$.
Model Description

- Market economy in a finite number of states, \( \mathcal{X} = \{1, 2, \ldots, k\} \).

- Model market dynamics by an irreducible CTMC, \( X = \{X_t\}_{t \geq 0} \).
  Transition rates

\[
P(X_{t+\delta t} = j \mid X_t = i) = \lambda_{ij} \delta t + o(\delta t), \ i \neq j
\]

\[
(\lambda_{ij} \geq 0, \ i \neq j, \ \lambda_{ii} = - \sum_{j=1}^{k} \lambda_{ij}).
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- Asset process

  \[ dA_t = A_t \left[ \left( \mu(X_t) - \kappa(X_t) \right) dt + \sigma(X_t) dW_t \right], \ A_0 > 0. \]
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- Amount in a money market account,
  
  \[ B_t = e^{\int_0^t r(X_u) \, du}, \quad B_0 = 1. \]
Model I

- Merton’s classical model (Merton, 1974) in a Markov modulated market.
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Default occurs iff, $A_T < K$. 

Total payoff to equity holders at maturity

$$E(T, A_T, X_T) = (A_T - K) + \max(0, A_T - K).$$

Value of the defaultable bond at maturity

$$D(T, A_T, X_T) = \min(K, A_T) = K - (K - A_T).$$
Model I

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- Default occurs iff, $A_T < K$.

- Total payoff to equity holders at maturity

$$E(T, A_T, X_T) = (A_T - K)^+ := \max(A_T - K, 0).$$

- Value of the defaultable bond at maturity

$$D(T, A_T, X_T) = \min(K, A_T) = K - (K - A_T)^+. $$
Model II

- Merton’s first passage model (Black and Cox, 1976) in a Markov modulated market.
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- Default time: $\tau = \min(\tau_1, \tau_2)$.

  $$\tau_1 = \begin{cases} T & \text{if } A_T < K \\ \infty & \text{otherwise.} \end{cases}$$

  $$\tau_2 = \inf \left\{ t \in (0, T) \mid A_t < J \right\}, \quad J < K.$$
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  \[ \tau_2 = \inf \left\{ t \in (0, T) \mid A_t < J \right\}, \quad J < K. \]

- \( E(T, A_T, X_T) = (A_T - K)^+ \mathbf{1}\{\hat{M}_T \geq J\}, \quad \hat{M}_t = \min_{0 \leq s \leq t} A_s. \)

- \( D(T, A_T, X_T) = K - (K - A_T)^+ + (A_T - K)^+ \mathbf{1}\{\hat{M}_T < J\}. \)
Default Behavior

$A_0$ : Initial Asset

K : Face Value ; J : Barrier Level

T (Maturity)
Default Behavior

\[ A_0 : \text{Initial Asset} \]
\[ K : \text{Face Value} ; J : \text{Barrier Level} \]
Default Behavior

$A_0$: Initial Asset

$K$: Face Value; $J$: Barrier Level
Default Behavior

$A_o$ : Initial Asset

$K$ : Face Value ; $J$ : Barrier Level
Approaches to Incomplete Market

1. Completing the market by introducing additional securities (Duffie and Huang, 1985).

2. Quadratic Hedging (Föllmer and Schweizer, 1991).

3. Esscher transform (Gerber and Elias, 1994).
Assume at time $T$, $H \in L^2(\Omega, \mathcal{F}_T, P)$.
- $\xi_t$: amount invested in asset $A_t$.
- $\eta_t$: amount invested in $B_t$. 
Quadratic Hedging in Incomplete Market

- Assume at time $T$, $H \in L^2(\Omega, \mathcal{F}_T, P)$.
  $\xi_t$ : – amount invested in asset $A_t$.
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- Consider a strategy $\pi = \{\pi_t\}_{0 \leq t \leq T} = \{\xi_t, \eta_t\}_{0 \leq t \leq T}$,
Quadratic Hedging in Incomplete Market

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- Consider a strategy $\pi = \{\pi_t\}_{0 \leq t \leq T} = \{\xi_t, \eta_t\}_{0 \leq t \leq T}$,

- Assume $\xi = \{\xi_t\}_{0 \leq t \leq T}$, predictable, and

$$E \left[ \int_0^T \xi_t^2 \sigma^2(X_t) A_t^2 \, dt + \left( \int_0^T |\xi_t| |\mu(X_t) - \kappa(X_t)| \, dt \right)^2 \right] < \infty.$$
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$$E \left[ \int_0^T \xi_t^2 \sigma^2(X_t) A_t^2 \, dt + \left( \int_0^T |\xi_t| |\mu(X_t) - \kappa(X_t)| \, dt \right)^2 \right] < \infty.$$

$\eta = \{\eta_t\}_{0 \leq t \leq T}$ is an adapted process, and $E\eta_t^2 < \infty$, $0 \leq t \leq T$. 
Few Terminologies

- Discounted value of portfolio (under $\pi$), $V_t^*(\pi) = \xi_t A_t^* + \eta_t$. 
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- Discounted cost process,

$$\tilde{C}_t(\pi) := V_t^*(\pi) - \int_0^t \xi_u dA_u^* - \int_0^t \xi_u \kappa(X_u) A_u^* du, \ 0 \leq t \leq T.$$
Few Terminologies

- **Discounted value of portfolio (under $\pi$),** \( V^*_t(\pi) = \xi_t A^*_t + \eta_t \).

- **Discounted cost process,**
  \[
  \tilde{C}_t(\pi) := V^*_t(\pi) - \int_0^t \xi_u \, dA^*_u - \int_0^t \xi_u \, \kappa(X_u) \, A^*_u \, du, \quad 0 \leq t \leq T.
  \]

- **Residual risk,**
  \[
  R_t(\pi) := E \left[ \left( \tilde{C}_T(\pi) - \tilde{C}_t(\pi) \right)^2 \bigg| \mathcal{F}_t \right].
  \]
Few Terminologies

- Discounted value of portfolio (under \( \pi \)), \( V^*_t(\pi) = \xi_t A^*_t + \eta_t \).

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- Residual risk,

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R_t(\pi) := E\left[ \left( \tilde{C}_T(\pi) - \tilde{C}_t(\pi) \right)^2 \bigg| \mathcal{F}_t \right].
\]

- A hedging strategy \( \pi^* \) is risk-minimizing if

\[
0 \leq R_t(\pi^*) \leq R_t(\pi), \quad 0 \leq t \leq T,
\]

for any other admissible strategy \( \pi \).
Definition (Locally Risk Minimizing Strategy)

An admissible strategy $\pi^*$ is said to be *locally risk minimizing* if the corresponding discounted cost $\{\tilde{C}_t(\pi^*)\}$ is a square integrable martingale orthogonal to $\{M_t\}$, where

$$M_t := \int_0^t \sigma(X_u) A_u^* \, dW_u.$$
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An admissible strategy $\pi^*$ is said to be *locally risk minimizing* if the corresponding discounted cost $\{\tilde{C}_t(\pi^*)\}$ is a square integrable martingale orthogonal to $\{M_t\}$, where

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Definition (Minimal Martingale Measure)

An EMM $P'$ equivalent to $P$ is said to be *minimal* if $P' \equiv P$ on $\mathcal{F}_0$ and if any square integrable $P$-martingale which is orthogonal to $\{M_t\}$ under $P$ remains a martingale under $P'$. 
Föllmer-Schweizer decomposition

- Existence of an optimal strategy for hedging $H^{FS,1991}$ existence of a decomposition of the discounted contingent claim $B_T^{-1}H$,

$$B_T^{-1}H = H_0 + \int_0^T \xi^H_u \left( dA^*_u + \kappa(X_u) A^*_u \, du \right) + L^H_T,$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P)$ and $L^H = \{L^H_t\}_{0 \leq t \leq T}$ is a square integrable martingale orthogonal to $\{M_t\}$. 
Föllmer-Schweizer decomposition

- Existence of an optimal strategy for hedging $H^{FS,1991}$ existence of a decomposition of the discounted contingent claim $B_T^{-1} H$

$$B_T^{-1} H = H_0 + \int_0^T \xi_u^{H^*} \left( dA_u^* + \kappa(X_u) A_u^* \, du \right) + L_t^{H^*},$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P)$ and $L_t^{H^*} = \{ L_t^{H^*} \}_{0 \leq t \leq T}$ is a square integrable martingale orthogonal to $\{ M_t \}$.

- Corresponding optimal strategy $\pi_t^* = (\xi_t^*, \eta_t^*)$

$$\xi_t^* := \xi_t^{H^*}, \quad \eta_t^* := V_t^* - \xi_t^* A_t^*,$$

with

$$V_t^* := H_0 + \int_0^t \xi_u^{H^*} \left( dA_u^* + \kappa(X_u) A_u^* \, du \right) + L_t^{H^*},$$

and

$$\tilde{C}_t(\pi^*) = H_0 + L_t^{H^*}.$$
The Minimal Martingale Measure

- The *market price of risk* in regime $i$ ($i = 1, 2, ..., k$),

$$
\gamma(i) := \frac{\mu(i) - r(i)}{\sigma(i)}.
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The Minimal Martingale Measure

- The *market price of risk* in regime $i$ ($i = 1, 2, ..., k$),

$$\gamma(i) := \frac{\mu(i) - r(i)}{\sigma(i)}.$$ 

- Let

$$\rho_t := \exp \left(- \int_0^t \gamma(X_u) \, dW_u - \frac{1}{2} \int_0^t \gamma(X_u)^2 \, du \right), \quad 0 \leq t \leq T,$$

and

$$dP^* := \rho_T \, dP.$$
The Minimal Martingale Measure

- The *market price of risk* in regime $i$ ($i = 1, 2, \ldots, k$),

\[ \gamma(i) := \frac{\mu(i) - r(i)}{\sigma(i)} . \]

- Let

\[ \rho_t := \exp \left( - \int_0^t \gamma(X_u) \, dW_u - \frac{1}{2} \int_0^t \gamma(X_u)^2 \, du \right), \quad 0 \leq t \leq T, \]

and

\[ dP^* := \rho_T \, dP. \]

**Lemma**

The EMM $P^*$ is the unique minimal martingale measure.
Non-defaultable Bond Price

\[ B(t, T, X_t) := E^* \left[ e^{-\int_t^T r(X_u)du} \mid X_t \right]. \]
Non-defaultable Bond Price

\[ B(t, T, X_t) := E^* \left[ e^{-\int_t^T r(X_u)du} \right | X_t]. \]

\[ \tilde{B}(t, T) := [B(t, T, 1), \ldots, B(t, T, k)]'. \]
Non-defaultable Bond Price

\[ B(t, T, X_t) := E^* \left[ e^{- \int_t^T r(X_u) du} | X_t \right]. \]

\[ \tilde{B}(t, T) := [B(t, T, 1), \ldots, B(t, T, k)]'. \]

**Theorem (Elliott, Hunter and Jamieson, 2001)**

The quantity \( \tilde{B}(t, T), t \in (0, T), \) satisfies

\[ \tilde{B}(t, T) = \exp \left( (\Lambda - R)(T - t) \right) 1, \]

where \( 1 \) is a \( k \times 1 \) vector with all entries equal to one and \( \Lambda = [\lambda_{ij}] \) is the generator matrix of the MC \( \{X_t\}_{t \geq 0}. \)
Contingent claim, $H_1 = K - (K - A_T)^+$. 
Pricing in Model I

- Contingent claim, \( H_1 = K - (K - A_T)^+ \).

- Let

\[
\varphi(t, A_t, X_t) := E^* \left[ e^{-\int_t^T r(X_u) du} \left( K - (K - A_T)^+ \right) \right| \mathcal{F}_t].
\]

- Define

\[
\mathcal{A}_1 = \left\{ f \bigg| f \in C([0, T] \times \mathbb{R}) \cap C^{1,2}((0, T) \times \mathbb{R}), \right.
\]

\[
\bigg. f \text{ has at most polynomial growth} \bigg\}.
\]
\( \varphi(t, A_t, X_t) \) satisfies (Feynmann-Kac),

\[
\frac{\partial \varphi(t, s, i)}{\partial t} + \frac{1}{2} \sigma(i)^2 s^2 \frac{\partial^2 \varphi(t, s, i)}{\partial s^2} + \left( r(i) - \kappa(i) \right) s \frac{\partial \varphi(t, s, i)}{\partial s} \\
+ \sum_{j=1}^{k} \lambda_{ij} \varphi(t, s, j) = r(i) \varphi(t, s, i),
\]

\( \varphi(T, s, i) = K - (K - s)^+, \ i = 1, 2, \ldots, k. \)
Pricing in Model I (Contd.)

- \( \varphi(t, A_t, X_t) \) satisfies (Feynmann-Kac),

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\frac{\partial \varphi(t, s, i)}{\partial t} + \frac{1}{2} \sigma(i)^2 s^2 \frac{\partial^2 \varphi(t, s, i)}{\partial s^2} + \left( r(i) - \kappa(i) \right) s \frac{\partial \varphi(t, s, i)}{\partial s} \\
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\]

\[
\varphi(T, s, i) = K - (K - s)^+, \quad i = 1, 2, \ldots, k.
\]

- \( \{ \varphi^{(1)}(t, s, i), \quad i = 1, 2, \ldots, k \} \) be the unique solution to the above system of PDEs in the class of functions belonging to the set \( A_1 \).
Theorem (Bond Price, Model I)

1. \( \{ \varphi^{(1)}(t, s, i), \ i = 1, 2, \ldots, k \} \) is the locally risk minimizing price of the defaultable bond.

2. An optimal strategy \( \pi^* = \{ \xi^*_t, \eta^*_t \} \) is given by

\[
\xi^*_t = \frac{\partial \varphi^{(1)}(t, A_t, X_{t-})}{\partial s},
\]

\[
\eta^*_t = V^*_t - \xi^*_t A^*_t.
\]

3. The residual risk at time \( t \) is given by

\[
R_t(\pi^*) = E \left[ \int_t^T e^{-2 \int_0^u r(X_v) dv} \sum_{j \neq X_u} \lambda_{X_u j} \left( \varphi^{(1)}(u, A_u, j) - \varphi^{(1)}(u, A_u, X_u) \right)^2 du \mid \mathcal{F}_t \right].
\]
Contingent claim, \( H_2 = A_T - (A_T - K)^+ \mathbf{1}(\hat{M}_T \geq J) \).
Pricing in Model II

- Contingent claim, \( H_2 = A_T - (A_T - K)^+ \mathbf{1}(\hat{M}_T \geq J) \).

- Consider the system of PDEs

\[
\frac{\partial \varphi(t, s, i)}{\partial t} + \frac{1}{2} \sigma(i)^2 \ s^2 \ \frac{\partial^2 \varphi(t, s, i)}{\partial s^2} + \left( r(i) - \kappa(i) \right) \ s \ \frac{\partial \varphi(t, s, i)}{\partial s} \\
+ \sum_{j=1}^{k} \lambda_{ij} \ \varphi(t, s, j) = r(i) \ \varphi(t, s, i),
\]

\( \varphi(T, s, i) = s, \ i = 1, 2, \ldots, k. \)
Pricing in Model II

- Contingent claim, \( H_2 = A_T - (A_T - K)^+ 1(\hat{M}_T \geq J) \).

- Consider the system of PDEs

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+ \sum_{j=1}^{k} \lambda_{ij} \varphi(t, s, j) = r(i) \varphi(t, s, i),
\]

\( \varphi(T, s, i) = s, \ i = 1, 2, \ldots, k. \)

- \( \{ \varphi^{(1)}(t, s, i), i = 1, 2, \ldots, k \} \) be the unique solution in the class of functions belonging to the set \( \mathcal{A}_1 \).
System of PDEs on \( D^+ := \{(t, s, i) \in (0, T) \times (J, \infty) \times \mathcal{X}\} \),

\[
\frac{\partial \varphi(t, s, i)}{\partial t} + \frac{1}{2} \sigma(i)^2 \, s^2 \frac{\partial^2 \varphi(t, s, i)}{\partial s^2} + \left( r(i) - \kappa(i) \right) \, s \frac{\partial \varphi(t, s, i)}{\partial s} + \sum_{j=1}^{k} \lambda_{ij} \varphi(t, s, j) = r(i) \, \varphi(t, s, i),
\]

\[
\varphi(T, s, i) = (s - K)^+, \quad s > J, \quad i = 1, 2, \ldots, k,
\]

\[
\varphi(t, J, i) = 0, \quad i = 1, 2, \ldots, k.
\]
Pricing in Model II (Contd.)

- System of PDEs on $\mathcal{D}^+ := \{(t, s, i) \in (0, T) \times (J, \infty) \times \mathcal{X}\}$,

$$
\frac{\partial \varphi(t, s, i)}{\partial t} + \frac{1}{2} \sigma(i)^2 s^2 \frac{\partial^2 \varphi(t, s, i)}{\partial s^2} + \left( r(i) - \kappa(i) \right) s \frac{\partial \varphi(t, s, i)}{\partial s} \\
+ \sum_{j=1}^{k} \lambda_{ij} \varphi(t, s, j) = r(i) \varphi(t, s, i),
$$

$$
\varphi(T, s, i) = (s - K)^+, \ s > J, \ i = 1, 2, \ldots, k,
$$

$$
\varphi(t, J, i) = 0, \ i = 1, 2, \ldots, k.
$$

- Define $\mathcal{A}_2 := \left\{ f \ \bigg| \ f \in C(\overline{\mathcal{D}}^+) \cap C^{1,2}(\mathcal{D}^+) \right\}$.

- $\{\varphi^{(2)}(t, s, i), \ i = 1, 2, \ldots, k\}$ be the unique solution for functions in $\mathcal{A}_2$ with at most polynomial growth.
Recall that $\tau$ denotes the default time in Model II.

**Theorem (Bond Price, Model II)**

1. $\varphi^{(1)}(t, A_t, X_t) - \varphi^{(2)}(t, A_t, X_t) \ 1(\tau > t)$ is the locally risk minimizing price of the defaultable bond.

2. An optimal strategy $\pi^* = \{\xi^*_t, \eta^*_t\}$ is given by

$$
\xi^*_t = \frac{\partial \varphi^{(1)}(t, A_t, X_{t-})}{\partial s} - \frac{\partial \varphi^{(2)}(t, A_t, X_{t-})}{\partial s} \ 1(\tau > t),
$$

$$
\eta^*_t = V^*_t - \xi^*_t A^*_t.
$$

3. Residual risk can also be calculated as in Model I.
Credit Spread

**Definition (Credit Spread)**

Credit spread is the difference between the yield on a defaultable bond and the yield on an otherwise equivalent default-free zero coupon bond.

\[
\text{Credit spread} = -\frac{\log \left( \frac{D(t,A_t,X_t)}{B(t,T,X_t)} \right)}{T - t}, \quad 0 \leq t < T.
\]
Definition (Credit Spread)

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\]

Initial leverage ratio \((L) = \frac{K}{A_0}\).


**Spread in Model I and II**

- \( k = 3, \ L = 0.40. \)
Spread in Model I and II

- $k = 3$, $L = 0.40$.

- TPM of the embedded discrete time Markov chain

$$P_1 = \begin{bmatrix} 0 & 0.9 & 0.1 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0.9 & 0 \end{bmatrix}.$$
$k = 3, \ L = 0.40.$

TPM of the embedded discrete time Markov chain

$$P_1 = \begin{bmatrix} 0 & 0.9 & 0.1 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0.9 & 0 \end{bmatrix}.$$ 

Holding time parameters are $\nu_1 = \nu_2 = \nu_3 = 6,$
Spread in Model I and II

- $k = 3, L = 0.40$.

- TPM of the embedded discrete time Markov chain

$$P_1 = \begin{bmatrix}
0 & 0.9 & 0.1 \\
0.5 & 0 & 0.5 \\
0.1 & 0.9 & 0
\end{bmatrix}.$$ 

- Holding time parameters are $\nu_1 = \nu_2 = \nu_3 = 6$,

- Other parameters

$$\left[r(i), \sigma(i), \kappa(i)\right] = \begin{cases}
(0.03, 0.1, 0.01), & \text{if } i = 1 \\
(0.07, 0.3, 0.04), & \text{if } i = 2 \\
(0.11, 0.5, 0.07), & \text{if } i = 3
\end{cases}. $$
Spread Comparison in Model I

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![Spread in Model I](image_url)

- **Merton**
- **State 1**
- **State 2**
- **State 3**
Spread Comparison in Model II

Spread for Model II

- Merton
- State 2

Maturity (in years)

Spread over risk free rate (bp)

- Merton
- State 2
Short Spreads

Short Spread Comparisons for Model I

Maturity (in days)
Spread over risk free rate(bp)

Merton
State 2
Example for Business Cycle Effects

- $k = 4$, $L = 0.40$. 
Example for Business Cycle Effects

- $k = 4$, $L = 0.40$.

\[
P_2 = \begin{bmatrix}
0 & 0.2 & 0.8 & 0 \\
0.8 & 0 & 0.2 & 0 \\
0 & 0.2 & 0 & 0.8 \\
0 & 0.8 & 0.2 & 0 \\
\end{bmatrix},
\]
Example for Business Cycle Effects

- $k = 4, L = 0.40.$

- $P_2 = \begin{bmatrix}
0 & 0.2 & 0.8 & 0 \\
0.8 & 0 & 0.2 & 0 \\
0 & 0.2 & 0 & 0.8 \\
0 & 0.8 & 0.2 & 0 \\
\end{bmatrix}$,

- $\nu_1 = 2, \nu_2 = 6, \nu_3 = 6, \nu_4 = 2,$
Example for Business Cycle Effects

- $k = 4$, $L = 0.40$.

- $P_2 = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0.8 & 0.2 & 0 \end{bmatrix}$,

- $\nu_1 = 2$, $\nu_2 = 6$, $\nu_3 = 6$, $\nu_4 = 2$,

- $[r(i), \sigma(i), \kappa(i)] = \begin{cases} (0.08, 0.6, 0.06), & \text{if } i = 1 \\ (0.07, 0.3, 0.03), & \text{if } i = 2 \\ (0.07, 0.4, 0.04), & \text{if } i = 3 \\ (0.06, 0.1, 0.01), & \text{if } i = 4. \end{cases}$
Business Cycle Effects

Effect of Business Cycle

- State 2
- State 3
Example for Rare state effects

- $k = 6, \ L = 0.40.$
Example for Rare state effects

- \( k = 6, \ L = 0.40 \).

\[
P_3 = \begin{bmatrix}
0 & 0.2 & 0.8 & 0 & 0 & 0 \\
0.8 & 0 & 0.18 & 0 & 0.02 & 0 \\
0 & 0.2 & 0 & 0.8 & 0 & 0 \\
0 & 0.8 & 0.2 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0 & 0 & 0.2 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\]
Example for Rare state effects

- $k = 6, L = 0.40$.

- $P_3 = \begin{bmatrix}
0 & 0.2 & 0.8 & 0 & 0 & 0 \\
0.8 & 0 & 0.18 & 0 & 0.02 & 0 \\
0 & 0.2 & 0 & 0.8 & 0 & 0 \\
0 & 0.8 & 0.2 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0 & 0 & 0.2 \\
0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix}$,

- $\nu_i = 6, \ i = 1, 2, 3, 4, \ \nu_5 = \nu_6 = 2$.

- Rare states: State 5 and 6.
Example for Rare state effects (Contd.)

- Stationary distribution

\[ \pi = \left[ 0.219, 0.274, 0.268, 0.214, 0.021, 0.004 \right]. \]

\[
\begin{align*}
\left[ r(i), \sigma(i), \kappa(i) \right] &= \begin{cases} 
(0.08, 0.4, 0.04), & \text{if } i = 1 \\
(0.06, 0.2, 0.02), & \text{if } i = 2 \\
(0.06, 0.3, 0.03), & \text{if } i = 3 \\
(0.04, 0.1, 0.01), & \text{if } i = 4 \\
(0.02, 0.7, 0.04), & \text{if } i = 5 \\
(0.01, 0.8, 0.06), & \text{if } i = 6.
\end{cases}
\end{align*}
\]
*Rare state effects*

**Figure:** Data 1:- State 2 after adding rare states, Data 2:- State 2 without the rare states
Future Research

- Analyzing regime switching effects in bond pricing for other credit risk models.

- Introducing regime switching in the default barrier level.

- Generalizing the models to semi-Markov modulated market.
Few References


Few References


Thank You!