On Einstein four-manifolds with S^1 -actions

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Abstract. We study closed Einstein 4-manifolds which admit S^1 actions of a certain type, i.e., warped products. In particular, we classify them up to isometry when the fixed point of the S^1 action satisfies certain natural geometric conditions. The proof uses the Bochner-Weitzenböck formula for 1-forms and the theory of minimal surfaces in 3-manifolds.

1. Introduction

A Riemannian n -manifold (N^n, h) is called Einstein if its Ricci tensor Ric_h is proportional to the metric h , i.e., $\text{Ric}_h = \lambda h$, where λ is a constant. If $n = 2$ or 3 , this condition implies that h has constant sectional curvature. If $n = 4$, the Einstein condition is much weaker but still has topological implications. It is not known if there is any obstruction to the existence of an Einstein metric when $n > 4$. In this work we restrict ourselves to $n = 4$.

The isometric classification of Einstein manifolds appears very difficult without further restrictions. Hence it is natural to assume that the isometry group of (N, h) is sufficiently “non-trivial”. Here “non-trivial” means that the dimension of the isometry group is sufficiently big. If this dimension is at least four, a complete classification is known in the compact case [1]. Classification in the three dimensional case is not complete, at least to the author’s knowledge, but important cases have been studied in [1] and [4].

If (N, h) is compact Einstein and we only assume that the dimension of the isometry group is non-zero, then it is known (cf. [2]) that either N is a flat 4-manifold or $\lambda > 0$. Since compact flat manifolds (in any dimension) are completely understood by Bieberbach’s theorem, we consider only the case $\lambda > 0$. The known simply connected examples with $\lambda > 0$ are S^4 , $S^2 \times S^2$, $\mathbb{C}P^2 \# k \overline{\mathbb{C}P}^2$ where $k = 0, 1, 3$. The metrics on these manifolds, upto homothety, are as follows: the first two have

the standard metrics, and the last has the Fubini-Study metric for $k = 0$, the Page metric [11] for $k = 1$ and the Tian-Yau metric [14] for $k = 3$. Note that the Kähler-Einstein metrics of Tian-Yau on $\mathbb{C}P^{2\#k}\overline{\mathbb{C}P}^2$ for $k = 4, \dots, 8$ have zero-dimensional isometry groups since $\mathbb{C}P^{2\#k}\overline{\mathbb{C}P}^2$, with the appropriate complex structures, does not admit global holomorphic vector fields for $k = 4, \dots, 8$.

We are interested in classifying, upto isometry, a sub-class of the compact Einstein 4-manifolds with isometric S^1 -actions.

To define this sub-class, first consider the class of Riemannian 4-manifolds (N, h) (not necessarily Einstein) with an isometric S^1 -action which satisfy the following two conditions:

- (a) The action is semi-free, i.e., the stabilizer of any point is either the whole group or just the identity.
- (b) If F denotes the fixed-point set of the action, then there is an isometry

$$\phi : (N \setminus F, h) \rightarrow (U \times S^1, g + u^2 d\theta^2)$$

where (U, g) is a Riemannian 3-manifold, $(S^1, d\theta^2)$ is the circle of radius 1, u is a smooth positive function on U . Here $\cong$ denotes an isometric equivalence which conjugates the S^1 -actions on $U \times S^1$ and on $N \setminus F$. The S^1 -action on $U \times S^1$ is by rotation on the S^1 factor.

In this paper we will refer to (N, h) , following the terminology of General Relativity, as *static*. We will also refer to the action as a *static* action. In Riemannian geometry (N, h) is called a *warped product*, with 1-dimensional fibres.

The goal of this paper is to classify the set of static Einstein 4-manifolds under certain conditions. To motivate these conditions let us consider the two known examples of compact simply-connected static Einstein 4-manifolds.

We will denote the standard Riemannian n -sphere of radius r in $\mathbb{R}^{n+1}$ by $(S^n, g_{S^n(r)})$. We normalize h so that $\lambda = 3$, this being the value of λ for $(S^4, g_{S^4(1)})$. Then the known examples are: $(S^4, g_{S^4(1)})$ and $(S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$.

To describe these more explicitly, we clarify some notations: if X is a topological space and $\sim$ an equivalence relation on X , then $\frac{X}{\sim}$ will denote the space of equivalence classes with the quotient topology.

- (i) $(S^4, g_{S^4(1)})$ is isometric to (N, h) , where

$$N = \frac{[0, \frac{\pi}{2}] \times S^2 \times S^1}{\sim}, \quad h = dt^2 + \sin^2(t)g_{S^2(1)} + \cos^2(t)d\theta^2.$$

Here the equivalence classes are $\{0\} \times S^2 \times \{y\}$, $\{\frac{\pi}{2}\} \times \{x\} \times S^1$, and $\{(t, x, y)\}$ for $t \neq 0, \frac{\pi}{2}$. It is easy to check that h is actually an analytic metric even at $t = 0$ and $t = \frac{\pi}{2}$. The isometry is given by sending (t, x, y) in N to $(\sin(t)x, \cos(t)y)$ in $\mathbb{R}^3 \times \mathbb{R}^2$. There is a S^1 -action on N given by rotation on the last factor.

The fixed point set $F = \frac{\{\frac{\pi}{2}\} \times S^2 \times S^1}{\sim}$. The action is static since

$$(N \setminus F, h) = ((0, \frac{\pi}{2}) \times S^2 \times S^1, dt^2 + \sin^2(t)g_{S^2(1)} + \cos^2(t)d\theta^2).$$

In this case F is isometric to $(S^2, g_{S^2(1)})$.

(ii) $(S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$ is isometric to (N, h) , where

$$N = \frac{[0, \pi] \times S^2 \times S^1}{\sim}, \quad h = dt^2 + g_{S^2(a)} + a^2 \sin^2\left(\frac{t}{a}\right) d\theta^2$$

where $a = \frac{1}{\sqrt{3}}$. Here the equivalence classes are $\{0\} \times \{x\} \times S^1$, $\{\pi\} \times \{x\} \times S^1$ and $\{(t, x, y)\}$ where $t \neq 0, \pi$. Again h is an analytic metric even at $t = 0$ and $t = \pi$. Here the isometry is given by sending (t, x, y) in N to $(x, \cos(t), \sin(t)y)$ in $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^2$. The S^1 -action on N is by rotation on the last factor.

The fixed point set $F = \frac{\{0\} \times S^2 \times S^1 \cup \{\pi\} \times S^2 \times S^1}{\sim}$. The action is static since

$$(N \setminus F, h) = ((0, \pi) \times S^2 \times S^1, dt^2 + g_{S^2(a)} + a^2 \sin^2\left(\frac{t}{a}\right) d\theta^2).$$

Here F is isometric to the disjoint union of two copies of $(S^2, g_{S^2(a)})$.

For a general static Riemannian 4-manifold, we prove that F is a disjoint union of totally geodesic closed surfaces. In the Einstein case, this geometry is important for determining the isometry type of (N, h) . In fact, our main result is that if the Gaussian curvature of F satisfies certain bounds, then the Einstein manifold (N, h) is isometric to one of the two examples above.

To motivate the conditions of our main theorem, recall that

- i) when $(N, h) = (S^4, g_{S^4(1)})$, F is connected and has Gaussian curvature 1,
- ii) when $(N, h) = (S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$, F is disconnected and has Gaussian curvature 3.

Theorem 1.1. *Let (N, h) be a closed, simply-connected, static Einstein 4-manifold, normalized so that $\text{Ric}_h = 3h$. Then the fixed-point set F of the action consists of totally geodesic 2-spheres with Gaussian curvature $K \geq 1$.*

If $K = 1$ at every point of F , then (N, h) is isometric to $(S^4, g_{S^4(1)})$.

If $K \leq 3$ at every point of F and F has more than one component, then (N, h) is isometric to $(S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$.

The technical reason for the assumption of disconnectedness of F in the last part of the above theorem is the following: it guarantees that the orbit space (which is a 3-manifold with boundary, see Lemma 2.5) of the action is not a handlebody (i.e., a compact 3-manifold whose boundary is a connected surface). This will ensure the existence of a stable minimal surface in the orbit space, which, in turn, will imply rigidity when combined with the curvature assumption. For details, see Section 4.

The paper is organized into three sections. In the first part of Section 2, some facts about static Riemannian manifolds are proved. In the second part, static Einstein manifolds are treated. In particular, the equations on the 3-manifold (M, g) corresponding to the Einstein equation on (N, h) are introduced. In Section 3 we derive the curvature bound $K \geq 1$ for the fixed point set and prove that $K = 1$ implies $(N, h) \cong (S^4, g_{S^4(1)})$. The main tool used here is the Bochner-Weitzenböck

formula for 1-forms. In Section 4, the case of a disconnected fixed point set is studied. Under the bound $K \leq 3$, we prove that $(N, h) \cong (S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$. Here the main tool is the theory of minimal surfaces in 3-manifolds.

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2. Preliminary results

2.1. Static Riemannian 4-manifolds

In this subsection we prove some general results about closed static analytic Riemannian 4-manifolds (N, h) without assuming the Einstein condition. Throughout the paper, a manifold will be called “closed” if it is compact and without boundary. A Riemannian manifold (N, h) will be called “analytic” if geodesic normal coordinates form a system of analytic charts and a quantity will be called “analytic” if it is real-analytic in geodesic normal coordinates. Recall that

$$(N \setminus F, h) \cong (U \times S^1, g + u^2 d\theta^2),$$

where $F = \{x \in N : gx = x \text{ for any } g \in S^1\}$ is the fixed-point set.

We first note:

Proposition 2.1. *A static S^1 -action on a Riemannian 4-manifold cannot have isolated fixed points.*

Proof. Let (N, h) be a static 4-manifold. Suppose p in N is an isolated fixed point. Let B be an open geodesic ball with center p which is homeomorphic to an Euclidean ball. B is invariant under the isometric S^1 -action. By staticity, $B \setminus \{p\}$ is homeomorphic to $S^1 \times O$, where O is an open subset of U . This is impossible, since $B \setminus \{p\}$ is simply connected. $\square$

Remark. Note that a (N, h) can admit a S^1 -action for which it is static and another S^1 -action for which it is not. For example, S^1 acts on $(S^4, g_{S^4(1)})$ by $(\alpha, (z_1, z_2, t)) \mapsto (\alpha z_1, \alpha z_2, t)$ where $\alpha \in S^1 \subset \mathbb{C}$ and $(z_1, z_2, t) \in \mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$. This action is not static because it has isolated fixed points, which cannot happen for static actions by Proposition 2.1. But S^4 also admits a static action, as seen in the introduction.

Corollary 2.2. *The fixed point set of a static S^1 -action on a closed Riemannian 4-manifold consists of a disjoint union (possibly empty) of closed surfaces.*

Proof. This follows from the following basic fact from Kobayashi [9].
The connected components of the fixed point set of an isometric and semi-free S^1 -action on a Riemannian manifold are totally geodesic submanifolds of even co-dimensions.

In our case F has to consist of points and surfaces. Combining this with Proposition 2.1 implies the result. $\square$

Our analysis of the closed 4-manifold (N, h) is actually carried out on the metric completion M of the 3-manifold (U, g) . To this end, we prove the next theorem. We need to work only with analytic Riemannian manifolds because Einstein manifolds are analytic (cf. [5]).

Proposition 2.3. *Let (N, h) be a static, analytic Riemannian 4-manifold with $(N \setminus F, h) \cong (U \times S^1, g + u^2 d\theta^2)$ and let M denote the metric completion of (U, g) . Then*

- i) *M is a compact analytic 3-manifold with boundary and $\partial M = M \setminus U$.*
- ii) *The metric g extends analytically to M .*
- iii) *The function u extends analytically to M .*

Proof. For notational simplicity we identify (p, θ) with $\phi(p, \theta)$, where $\phi : (N \setminus F, h) \rightarrow (U \times S^1, g + u^2 d\theta^2)$ is the isometry in the definition of a static action. U will be identified with a $U \times \{\theta_0\}$ for some fixed θ_0 . Hence M will be identified with the closure of $U \times \{\theta_0\}$ in N .

We start by describing U intrinsically in terms of N and the S^1 action. Consider the distribution of tangent 3-planes on $N \setminus F$ defined by the orthogonal complement of the Killing field X of the action (note that $F = \{p : X_p = 0\}$). It is clear that the staticity of the action implies

- (a) that this distribution is integrable and, in fact, the integral submanifolds are exactly $U \times \{\theta\}$.
- (b) each integral submanifold is totally geodesic.

Our identification of U with a $U \times \{\theta_0\}$ for some fixed θ_0 amounts to identifying U with a fixed integral submanifold.

With this description the analyticity of g and u on U is seen as follows: since isometries of an analytic metric are analytic, so are X and $|X|^2$. From this it follows that the distribution defined above and its integral submanifolds are analytic. From (a) and (b) above, it is clear the metric g and the function $u(p) = |X|(p, \theta)$ are analytic on U .

Now we analyze the structure of M . It is clear that $M = U \cup F$, as sets. The analytic manifold structure at F and the regularity of g and u at F are best seen by considering the normal exponential map $\exp$. So let T be a tubular neighbourhood of $B \subset F$, where B is such that the normal trivial bundle is trivial on B and $\exp$ is a diffeomorphism. Using the S^1 symmetry we can find geodesic normal coordinates $(p, t, \theta) \in F \times (0, \epsilon) \times S^1$ for $T \setminus F$. In this description, an integral submanifold is identified with the set $\theta = \text{constant}$ (by staticity) and hence the analytic manifold structure extends to F . So the induced metric on M , which extends g on U is analytic.

The regularity of u on F is slightly delicate because $u(p) \rightarrow 0$ as p approaches F . We look at the metric h in the coordinates above:

$$h = dt^2 + u(p, t)^2 d\theta^2 + a(p, t),$$

where $a(p, t)$ is the induced metric on the surfaces $t = \text{constant}$.

Extend u to $t = 0$, i.e. to F , by setting $u = 0$ there (this is forced by continuity). It is clear that the analyticity of $u(t, p)$ on M needs to be verified only

in the t variable at $t = 0$. Now, as the surface $p = \text{constant}$ is analytic, so is the induced metric on it, $dt^2 + u(p, t)^2 d\theta^2$. It is easy to see (after changing the (t, θ) coordinates to $(x = t \cos \theta, y = t \sin \theta)$) that this implies the analyticity of the function u at $t = 0$. In fact, we have the following elementary fact (cf. [12]):

Suppose g is a smooth (analytic) metric on $\mathbb{R}^2$ and $g = dt^2 + u(t)^2 d\theta^2$ on $\mathbb{R}^2 \setminus 0$. Assume that $u : [0, \infty) \rightarrow [0, \infty)$ is a continuous non-negative function which is smooth (analytic) and positive on $(0, \infty)$ and $u(0) = 0$. Then u is smooth (analytic) at $t = 0$ and $\frac{du}{dt}(0) = 1$.

We note for future reference that the last equality can be restated as follows: $|du| = 1$ on ∂M , where the norm is with respect to the metric g on M . We also note that since F is the boundary of an embedded 3-manifold M in N , the normal bundle of F is trivial (since its self-intersection number is zero). $\square$

The following observations from the preceding result and its proof will be used repeatedly. We use the fact that an Einstein metric is real-analytic in geodesic normal coordinates (cf. [5]).

Corollary 2.4. *The above extensions of u and g have the following properties:*

- i) *If h is an Einstein metric on N , then g is a real-analytic metric on M .*
- ii) *∂M is totally geodesic in M and is isometric to F . Here the metrics are induced from M and N respectively.*
- iii) *$u = 0$ and $|du| = 1$ on ∂M .*

Lemma 2.5. *Let N/S^1 be the orbit space of the static action, with the quotient topology. Then M is homeomorphic to N/S^1 .*

Proof. The homeomorphism $f : U \times S^1 \rightarrow N \setminus F$ gives rise to a homeomorphism $f_1 : U \rightarrow N \setminus F/S^1$. This is because f conjugates the S^1 -actions. By the identification of ∂M with F , it is seen that f_1 extends to a homeomorphism of M and N . $\square$

Remark 2.6. In fact, the following stronger statement is true in the above setting: N/S^1 with the distance function given by distance between orbits is isometric (as a metric space) to (M, g) . However, we will be using the orbit space interpretation of M only in the next lemma, which is purely topological and does not require the metric structure of N/S^1 .

The next result is about the topological structure of simply connected static 4-manifolds. We will freely use the topological and metric identification of F and ∂M in what follows.

Lemma 2.7. *If N is simply-connected, then*

- i) *M is simply-connected,*
- ii) *F consists of 2-spheres.*

Proof. First we prove that the simple-connectedness of N implies the simple-connectedness of M . Let α be any loop in M . By the embedding of M in N described in the proof of Proposition 2.3, α can be regarded as a curve in N . Now the homotopy

in N contracting α to a point can be composed with the canonical projection map π from N to N/S^1 . This gives a homotopy in N/S^1 contracting $\pi \circ \alpha$ to a point. Under the homeomorphism between N/S^1 and M described in the Lemma 2.5, we see that $\pi \circ \alpha$ corresponds to α . Hence α is homotopically trivial.

Now we prove that $H_1(M, \mathbb{Z}/2) = 0$ implies that $H_1(\partial M, \mathbb{Z}/2) = 0$. This follows from the long exact sequence for the pair $(M, \partial M)$:

$$\longrightarrow H_2(M, \partial M, \mathbb{Z}/2) \xrightarrow{\partial_*} H_1(\partial M, \mathbb{Z}/2) \xrightarrow{i_*} H_1(M, \mathbb{Z}/2) \rightarrow$$

By Poincaré Duality

$$H_2(M, \partial M, \mathbb{Z}/2) \cong H^1(M, \mathbb{Z}/2).$$

Also $H^1(M, \mathbb{Z}/2) \cong H_1(M, \mathbb{Z}/2) = 0$.

Hence ∂M consists of spheres or projective planes (note that $\partial M = F$ cannot be empty since N is simply connected). We rule out the latter as follows: Since N is simply-connected, it is orientable. Since the normal bundle of F in N is trivial (see the proof of Proposition 2.3), F is also orientable. By the identification of F and ∂M , we see that the latter is orientable. $\square$

2.2. Static Einstein 4-Manifolds

First, we note the following facts about static Einstein 4-manifolds.

Proposition 2.8. *If (N, h) is a closed, non-flat, static Einstein 4-manifold, then the scalar curvature of h is positive.*

Proof. This follows from the following classical theorem of Bochner:
Any Killing field on a closed Riemannian manifold of non-positive Ricci curvature is parallel.

In particular, if a Killing field is zero somewhere it has to be identically zero. As will be shown in the next lemma, the Killing field generated by the static action is zero somewhere. Since it is not identically zero, we conclude that the Ricci curvature cannot be non-positive. $\square$

Proposition 2.9. *If (N, h) is a closed, non-flat, static Einstein 4-manifold, then F is a non-empty disjoint union of surfaces.*

Proof. By Corollary 2.1, it is enough to prove that F is non-empty. If F is empty, then the Euler characteristic $\chi(N) = 0$. But the four-dimensional Gauss-Bonnet theorem for an Einstein manifold states that

$$\int_N |R|^2 = 8\pi^2 \chi(N),$$

where R is the Riemann curvature tensor. This implies that $R = 0$ i.e. N is flat. $\square$

We can now normalize the Ricci curvature of h , so that $\text{Ric}_h = 3h$. This is equivalent to the following two equations on M (cf. [2], Page 266).

$$\frac{D^2u}{u} = \text{Ric}_g - 3g \quad (1)$$

$$\Delta u = -3u, \quad (2)$$

where D^2u is the Hessian of u and Ric_g is the Ricci tensor of g . It should be noted that the sign of our Laplacian is opposite of the one in [2]. Equations (1) and (2) along with the restrictions $u = 0, |du| = 1$ on ∂M constitute a boundary value problem for the pair (g, u) consisting of the metric g and the non-negative function u on the 3-manifold with boundary M .

In the next two sections we will analyze some of the geometric consequences of this P.D.E. system. Here we give an example of a family of solutions to (1) and (2) and prove that they cannot satisfy both the boundary conditions $u = 0, |du| = 1$ on ∂M . Hence, they cannot arise from a *closed* static Einstein 4-manifold. This class of examples was also studied by J.Lafontaine in a different context (cf. [10]).

Proposition 2.10. *There exist $SO(2)$ invariant solutions, i.e. metrics g and nonnegative functions u , to (1) and (2) on $M = [0, T] \times S^2$, for certain $T > 0$. These satisfy $u = 0$ on ∂M but cannot satisfy $|du| = 1$ on ∂M .*

Proof. Let $M = I \times S^2(a)$ and $g = dt^2 + h(t)^2 g_{S^2(a)}$, where $I = [0, T]$ (the value T will be explained below), $S^2(a)$ is the 2-sphere of scalar curvature a . h satisfies the O.D.E which ensures that g has constant scalar curvature equal to 6, i.e.,

$$3h^2 + 2hh'' + (h')^2 = \frac{a}{2} \quad (3)$$

It can be checked that if $u = h'$, then (1) and (2) are satisfied. It can also be proved that there is an interval A such that if $a \in A$, then there is a periodic solution h with $h'(0) = 0, h''(0) = 1$. It follows that there exists T such that $h' > 0$ on $(0, T)$, $h'(T) = 0$ and $h > 0$ on $[0, T]$.

Hence $u = 0$ on $\partial M = \{0\} \times S^2(a) \cup \{T\} \times S^2(a)$.

Now we claim that there is no solution h such that $h' = 0$ and $|du|^2 = (h'')^2 = 1$ on ∂M . Suppose that h is such a solution. Consider the function $w = 3u^2 + |du|^2 = 3(h')^2 + (h'')^2$. We have

$$w' = 2h''(3h' + h''').$$

Differentiating Equation (3), we get

$$3h' + h''' = \frac{-2}{h} h' h''.$$

Therefore,

$$w' = \frac{-2}{h} h' (h'')^2 \leq 0.$$

Since $w = 1$ at $t = 0$ and $t = T$, we get $w' = 0$ everywhere. Hence, $h' = 0$ everywhere, which is impossible since h is a non-constant periodic function. $\square$

3. The curvature of F

3.1.

For the rest of the paper all the quantities considered will be on the 3-manifold (M, g) unless mentioned otherwise.

We first look at the two known examples of closed static Einstein 4-manifolds. These are the same as examples (i) and (ii) in the Introduction.

- (i) $(N, h) = (S^4, g_{S^4(1)})$. Here $(M, g) = (B^3, dt^2 + \sin^2(t)g_{S^2})$, where B^3 is the closed 3-ball, thought of as a quotient space $B^3 = \frac{[0, \frac{\pi}{2}] \times S^2}{\sim}$ ($\sim$ as in the Introduction).

Note that $u = \cos(t)$ and $u^2 + |du|^2 = 1$ on M .

- (ii) $(N, h) = (S^2 \times S^2, g_{S^2(a)} + g_{S^2(a)})$, where $a = \frac{1}{\sqrt{3}}$. Here $(M, g) = ([0, \pi] \times S^2, dt^2 + g_{S^2(a)})$.

Now $u = a \sin(\frac{t}{a})$ and $3u^2 + |du|^2 = 1$ on M .

In consideration of the examples above we will prove the following :

Lemma 3.1. *Let (M, g) be a compact Riemannian 3-manifold with boundary and $u : M \rightarrow [0, \infty)$ be a smooth nonnegative function with $u^{-1}(0) = \partial M$. Suppose g and u satisfy Equations (1) and (2). Then the function $f = u^2 + |du|^2$ on M cannot attain its global maximum in the interior of M unless it is constant and (M, g) has constant sectional curvature 1.*

Before proving the lemma we show how it implies the first part of Theorem 1.2.

Corollary 3.2. *Let (N, h) be a closed non-flat static Einstein 4-manifold. Then the Gaussian curvature $K(p)$ of F at p satisfies $K(p) \geq 1$ for all $p \in F$.*

Proof. We will use the identification of F and ∂M and prove the result for ∂M . In what follows, we will denote the gradient vector field of u by ∇u and we will use the equality $|\nabla u| = |du|$. If f attains its maximum in the interior, we are done since $K = 1$ in that case. So assume that f attains its global maximum on ∂M (note that Corollary 2.4 implies that $f = 1$ on ∂M) and since u attains its global minimum on ∂M , it follows that $\langle \nabla f, \nabla u \rangle(x) \leq 0$ for all $x \in M$ sufficiently close to p .

Let $X = \frac{\nabla u}{|\nabla u|}$.

We have

$$\begin{aligned} \langle \nabla f, \nabla u \rangle &= \langle 2u\nabla u + 2|du|\nabla|du|, \nabla u \rangle = 2u|du|^2 \left(1 + \frac{\langle \nabla|du|, \nabla u \rangle}{u|du|} \right) \\ &= 2u|du|^2 \left(1 + \frac{\langle \nabla|du|, X \rangle}{u} \right) = 2u|du|^2 \left(1 + \frac{X(X(u))}{u} \right). \end{aligned} \quad (4)$$

Therefore,

$$\frac{X(X(u))}{u} \leq -1. \quad (5)$$

Now $\langle X, X \rangle = 1$ implies that $\langle \nabla_X X, X \rangle = 0$. Recalling the definition of X , we then have $\nabla_X X(u) = 0$. Hence

$$\frac{XX(u)}{u} = \frac{D^2u(X, X)}{u}. \quad (6)$$

From (1) it follows that

$$\frac{-D^2u(X, X)}{u} = 3 - \text{Ric}_g(X, X) = K(x), \quad (7)$$

where $K(x)$ is the (extrinsic) sectional curvature of the surface $u^{-1}(u(x))$ at x . Note that we have used the fact g has scalar curvature 6 everywhere (which follows by taking trace on both sides of (1) and then using (2)) in the last equation.

Combining (5) and (7), we have $K(x) = \frac{-XX(u)}{u} \geq 1$. In the limit, as $x \rightarrow p$, we get $K(p) \geq 1$. Since $\partial M = u^{-1}(0)$ is totally geodesic, $K(p)$ is also the intrinsic (Gaussian) curvature of ∂M . $\square$

Now we prove Lemma 3.1. Our main tool is the Bochner-Weitzenböck formula for 1-forms:

$$\frac{1}{2} \Delta |du|^2 = |D^2u|^2 + \langle \nabla \Delta u, \nabla u \rangle + \text{Ric}_g(\nabla u, \nabla u). \quad (8)$$

Proof (of Lemma 3.1). Using (1) and (2) in (8),

$$\begin{aligned} \frac{1}{2} \Delta |du|^2 &= |D^2u|^2 - 3|\nabla u|^2 + \text{Ric}_g(\nabla u, \nabla u) \\ &= |D^2u|^2 + \frac{D^2u(\nabla u, \nabla u)}{u} \\ &= |D^2u|^2 + \frac{(\nabla u \nabla u(u) - \nabla_{\nabla u} \nabla u(u))}{u} \\ &= |D^2u|^2 + \frac{(\langle \nabla u, \nabla |du|^2 \rangle - \frac{1}{2} \nabla u \langle \nabla u, \nabla u \rangle)}{u} \\ &= |D^2u|^2 + \frac{1}{2u} \langle \nabla u, \nabla |du|^2 \rangle, \end{aligned} \quad (9)$$

which implies that

$$\frac{1}{2} \text{div} \left(\frac{\nabla |du|^2}{u} \right) = \frac{|D^2u|^2}{u}. \quad (10)$$

Now

$$\frac{1}{2} \Delta u^2 = u \Delta u + |du|^2 = -3u^2 + |du|^2, \quad (11)$$

where we have used (2).

This implies that

$$\frac{1}{2} \text{div} \left(\frac{\nabla u^2}{u} \right) = \frac{-3u^2}{u}. \quad (12)$$

Adding equations (10) and (12) we get

$$\frac{1}{2} \operatorname{div} \left(\frac{\nabla f}{u} \right) = \frac{1}{u} (|D^2 u|^2 - 3u^2). \quad (13)$$

But

$$|D^2 u|^2 \geq \frac{1}{3} (\Delta u)^2 = 3u^2.$$

Hence $\operatorname{div} \left(\frac{\nabla f}{u} \right) \geq 0$. Now, if $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is any smooth, non-negative, increasing function, we have

$$\operatorname{div} \left(\frac{\phi(f)}{u} \nabla f \right) = \phi(f) \operatorname{div} \left(\frac{\nabla(f)}{u} \right) + \frac{\phi'(f)}{u} |df|^2 \geq 0.$$

Suppose that the maximum of f is $m > 1$. Choose a smooth, non-negative, increasing ϕ so that $\phi(t) = 0$ if $t \leq 1$ and $\phi(t) = 1$ if $t \geq \frac{1+m}{2}$. Integrating the equation above, using Stokes' theorem and noting that the boundary integrals vanish because of the way we chose ϕ , we see that $\nabla f = 0$ and $|D^2(u)|^2 = 3u^2$ on an open set. By analyticity, $\nabla f = 0$ on M .

Hence $f = 1$ and $|D^2(u)|^2 = 3u^2$ on M .

To see (M, g) has constant sectional curvature 1, note that $|\operatorname{Ric}_g - 2g|^2 = 0$ from the above equation and (1). Since M is 3-dimensional, constant Ricci curvature 3 implies that g has constant sectional curvature 1. $\square$

Corollary 3.3. *Let (N, h) be a closed, simply connected, static Einstein 4-manifold. If $K = 1$ at every point of F , then (N, h) is isometric to $(S^4, g_{S^4(1)})$.*

Proof. Again we use the identification of F and ∂M and work with the latter. Using (1) and (2) in (13) gives

$$u|\operatorname{Ric}_g - 2g|^2 = \frac{1}{2} \operatorname{div} \left(\frac{\nabla f}{u} \right).$$

For ϵ small enough, let $M_\epsilon = M \setminus u^{-1}([0, \epsilon))$. M_ϵ is a manifold with boundary $u^{-1}(\epsilon)$. The outward normal to M_ϵ is $\nu = -\frac{\nabla u}{|\nabla u|}$.

Integrating the above equation on M_ϵ and using the divergence theorem, we get

$$\int_{M_\epsilon} u|\operatorname{Ric}_g - 2g|^2 dV = \frac{1}{2} \int_{u^{-1}(\epsilon)} \frac{\langle \nabla f, -\nabla u \rangle}{|\nabla u| u} dA \quad (14)$$

Now note that (4), (6) and (7) combine to give

$$\frac{\langle \nabla f, -\nabla u \rangle}{2|\nabla u| u} = \frac{1}{|\nabla u|} (K - 1) \quad (15)$$

Letting $\epsilon \rightarrow 0$ in (14), using $|\nabla u| = 1$ on ∂M (Corollary (2.4)) and (15), we get

$$\int_M u|\operatorname{Ric}_g - 2g|^2 dV = \int_{\partial M} (K - 1) dA.$$

Now, if $K = 1$ at every point of F , then the right integral is zero and hence $\text{Ric}_g = 2g$ on M . This again implies constant sectional curvature 1.

Next, we claim that ∂M is connected. Since ∂M consists of totally geodesic components, this follows from the following result of T. Frankel ([6]):

Let (M, g) be a complete Riemannian manifold with positive Ricci curvature. Then any two closed minimal hypersurfaces in M have to intersect.

As stated, Frankel's theorem applies to Riemannian manifolds (M, g) without boundary. However, an inspection of the proof (essentially an application of the second variation formula for the length functional) shows that if (M, g) is a Riemannian manifold with boundary such that g has positive Ricci curvature and ∂M is minimal, then ∂M is connected.

Hence $\partial M = S^2$. By Lemma 2.7, M is simply connected.

So (M, g) is simply-connected 3-manifold with totally geodesic boundary and constant sectional curvature 1. It is easy to see that this implies that (M, g) is isometric to a hemisphere. As in example (i) in the beginning of this section, we regard a hemisphere as a quotient space with a Riemannian metric as follows:

$$M = \frac{[0, \frac{\pi}{2}] \times S^2}{\sim}, \quad h = dt^2 + \sin^2(t)g_{S^2}.$$

To describe the structure of N , we solve for u .

From (1), we get $D^2u = -ug$. Hence, $D^2u(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -u$. i.e.,

$$\frac{\partial^2 u}{\partial t^2} = -u.$$

Therefore $u(t, x) = A(x) \cos(t) + B(x) \sin(t)$. By Corollary 2.4, u satisfies the boundary conditions $u = 0$ and $|du| = 1$ on ∂M . Hence $u(\frac{\pi}{2}, x) = 0$, $\frac{\partial u}{\partial t}(\frac{\pi}{2}, x) = 1$. So, we get $u(t, x) = \cos(t)$.

Therefore, with notation as in the Introduction,

$$N = \frac{[0, \frac{\pi}{2}] \times S^2 \times S^1}{\sim}, \quad h = dt^2 + \sin^2(t)g_{S^2} + \cos^2(t)d\theta^2.$$

Hence $(N, h) \cong (S^4, g_{S^4(1)})$. □

4. The Case of Disconnected F

The next theorem, which covers the remaining part of Theorem 2.4, is proved using techniques different from the above, namely the theory of minimal surfaces in 3-manifolds. This section relies heavily on the work of G. Galloway [7], M. Cai and G. Galloway [3] and W. Meeks, L. Simon and S.-T. Yau [15]. The arguments presented below are slight variants of those in [7], [3]. It should also be noted that an argument very similar to the one used in Case I below appears in a paper by S. Hwang [8]. In fact, Lemma 2 in Hwang's paper proves similar results for general semi-free S^1 actions, which may not be static. However, we give the proof for completeness.

Theorem 4.1. *Let N be a closed, simply-connected, static Einstein 4-manifold.*

If $K \leq 3$ and $F = \partial M$ has more than one component, then (N, h) is isometric to $(S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$.

Proof. An embedded surface in a 3-manifold will be called “one-sided” or “two-sided” depending on whether its normal bundle is trivial or non-trivial, respectively. Recall that a closed two-sided minimal surface Σ in a 3-manifold M is called *stable* if the second variation of the area functional is non-negative. i.e., $\frac{d^2}{dt^2} A(\Sigma_t)(0) \geq 0$ where $A(S)$ denotes the area of S and $\Sigma_t, t \in (-\epsilon, \epsilon)$ is any normal variation of Σ . A minimal surface Σ is called *locally area minimizing* if $A(\Sigma) \leq A(\Sigma_t)$ for all sufficiently small t . This condition implies stability.

The proof is based on the following two results:

- (A) (cf. [15], Theorem 1' and Proposition 1) Let M be a Riemannian 3-manifold-with-boundary which is not a handlebody. Assume that the mean curvature of the boundary is non-negative with respect to the outward normal. Then M contains a locally area minimizing surface, say Σ .
- (B) Let (M, g) satisfy (1) and (2). Then (M, g) cannot contain a locally area minimizing 2-sphere unless it is isometric to the Riemannian product $(S^2, g_{S^2(\frac{1}{\sqrt{3}})}) \times [0, \frac{\pi}{\sqrt{3}}]$.

(B) will immediately imply, by explicitly solving (1) and (2), that (N, h) is isometric to $(S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$.

(A) gives rise to two cases:

- (I) Σ is contained in the interior of M .
- (II) Σ intersects ∂M .

We will prove (B) for both the cases separately. Before this, we note the following:

First, in our case, since the components of ∂M are minimal surfaces (in fact, they are totally geodesic in by Proposition 2.4), case (II) actually implies that Σ is a component of ∂M . This follows from the maximum principle for minimal hypersurfaces [cf. [15], page 19].

Second, we can take Σ to be two-sided and homeomorphic to a 2-sphere, in our case. This is seen as follows: By Lemma 2.7, ∂M consists of 2-spheres. Fix a boundary component S . We can minimize the area (as in [15], Theorem 1) in the isotopy class of S . It is not true, in general, that the minimum is realized by a surface isotopic to S . However, it is proved in [15] that the minimum is realized by a disjoint collection of smooth closed embedded locally area minimizing surfaces S_i , $i = 1, \dots, k$ and the following relation holds:

$$\sum_{i \in \mathcal{A}} \frac{1}{2}(g_i - 1) + \sum_{i \in \mathcal{B}} g_i \leq \text{genus}(S)$$

where $g_i = \text{genus}(S_i)$, $\mathcal{A} = \{i : S_i \text{ is one-sided in } M\}$ and $\mathcal{B} = \{i : S_i \text{ is two-sided in } M\}$. Note that since $g_j \geq 1$ for all $j \in \mathcal{A}$, all the terms in the left side are non-negative. We claim that $\mathcal{A}$ is empty. This follows from a classical result

in topology (cf. [13]): *Any closed embedded hypersurface in a simply-connected manifold (of any dimension) has to be orientable.* This proves the claim, since M is simply-connected by the proof of Lemma 2.7. From the inequality above we see that $g_i = 0$ if i is in $\mathcal{B}$, since $\text{genus}(S) = 0$. Hence we get at least one embedded locally area minimizing two-sided S^2 .

Case I: We now prove (B) under the assumption of (I). The proof closely follows [7]. By (A) and the remark above we assume that we have a two-sided Σ (which is homeomorphic to S^2) in the interior of M . Hence $u > 0$ in a neighbourhood of Σ . Let ν be a unit normal vector field to Σ and ϕ a smooth positive function defined in a neighbourhood of Σ . Define the following variation of Σ : choose ϵ small enough and for each $t \in (-\epsilon, \epsilon)$, let Σ_t be the surface obtained by travelling in the direction of ν along geodesics in the metric $\phi^2 g$ for time t . Let

$H = H_t =$ mean curvature of Σ_t ,

$B = B_t =$ second fundamental form of Σ_t and

$\nu = \nu_t =$ normal to Σ_t .

We have the following evolution equation for B , in Gaussian normal coordinates (in $\phi^2 g$) t, x^1, x^2 . All the quantities are with respect to the metric g :

$$\frac{\partial b_{ij}}{\partial t} = \phi^{-1} R_{i3j3} + \phi b_i^m b_{mj} + \phi_{;ij}, \quad 1 \leq i, j \leq 2 \quad (16)$$

where R is the Riemann curvature tensor of (M, g) , b_{ij} and $\phi_{;ij}$ are the components of B and $D^2\phi$ respectively.

Taking the trace of equation (16) we get

$$\frac{\partial H}{\partial t} = \phi \text{Ric}_M(\nu, \nu) + \phi |B|^2 + \Delta_{\Sigma_t} \phi \quad (17)$$

where Δ_{Σ_t} is the Laplacian on Σ_t with the metric induced from M .

If $A(t)$ = the area of Σ_t , then the first variation formula gives

$$A'(t) = - \int_{\Sigma_t} \phi H dA \quad (18)$$

We now choose a specific variation given by taking $\phi = u$ and obtain an evolution equation for $\frac{H}{u}$. First,

$$\Delta u = \Delta_{\Sigma_t} u + D^2 u(\nu, \nu) - \frac{H}{u} \frac{\partial u}{\partial t}.$$

Using (2),

$$\Delta_{\Sigma_t} u = -3u - D^2 u(\nu, \nu) + \frac{H}{u} \frac{\partial u}{\partial t} \quad (19)$$

(1), (17) and (19) imply

$$\frac{\partial H}{\partial t} = u |B|^2 + \frac{H}{u} \frac{\partial u}{\partial t}.$$

Hence,

$$\frac{\partial}{\partial t} \left(\frac{H}{u} \right) = |B|^2.$$

Therefore $\frac{H}{u}$ is non-decreasing. Since Σ is minimal, $H_0 = 0$. Therefore $H_t \geq 0$ for $t \in [0, \epsilon)$. Hence, by (18), $A'(t) \leq 0$ for $t \in [0, \epsilon)$. If A' ever becomes strictly negative on $(0, \epsilon)$, then for some $t \in (0, \epsilon)$, $A(t) < A(0)$, contradicting the locally area minimizing property of Σ on $(0, \epsilon)$. Hence

$$\frac{\partial}{\partial t} \left(\frac{H}{u} \right) = |B|^2 = 0 \quad (20)$$

for all $t \in [0, \epsilon)$. By using the same variation, with respect to $-v$, we can conclude that (20) holds for all $t \in (-\epsilon, \epsilon)$

Equations (1), (16) and (20) imply that for all vectors $X, Y \in T_p \Sigma$

$$u^{-1} D_{\Sigma}^2 u(X, Y) = \text{Ric}_g(X, Y) - 3\langle X, Y \rangle = -\langle R(X, v)v, Y \rangle. \quad (21)$$

(21) and the definition of Ricci curvature imply that $\text{Ric}_g(e, e) = \frac{1}{2}(K + 3)$, where e is any unit vector tangent to Σ and K is the Gaussian curvature of Σ .

By polarization we get

$$\text{Ric}_g(X, Y) = \frac{1}{2}(K + 3)g_{\Sigma}(X, Y). \quad (22)$$

Combining this with (21) gives

$$D_{\Sigma}^2 u = \frac{1}{2}(K - 3)ug_{\Sigma}. \quad (23)$$

Galloway shows (cf. [7], page 61) that the only solutions to the above equation are $u = \text{constant}$ (on Σ) which occur when $K = 3$.

Hence it follows, for example by the De Rham decomposition theorem, that a neighbourhood of Σ is isometric to $(-\delta, \delta) \times S^2(a)$, for some $\delta > 0$. Here $a = \frac{1}{\sqrt{3}}$. By a continuation argument, we see that this product structure extends to all of M and we are done.

Case II: Assume that one of the boundary components, say Σ , of M is a locally area minimizing surface. We cannot directly use the preceding argument since $u = 0$ on Σ . We will prove that under our curvature assumptions, we again get a locally area minimizing surface in the interior and then we are back to Case I. We proceed as follows: Let v be a unit normal vector field on Σ . Choose ϵ small enough so that a tubular neighbourhood U of Σ is diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$, the diffeomorphism being the normal exponential map. As before, we will identify U and $(-\epsilon, \epsilon) \times \Sigma$. Σ_t , B_t and H_t will denote, as before, the surface $\{(t, x) : x \in \Sigma\}$, its second fundamental form and its mean curvature, respectively. B and H will denote B_0 and H_0 . If f is a function on U , $f^{(n)}$ will denote $\frac{\partial^n f}{\partial t^n}$.

By slightly modifying the proof of Cai and Galloway ([3], Theorem B and the Claim on page 391), we can prove

Let (M, g) be a smooth Riemannian 3-manifold with constant scalar curvature S . Suppose Σ is a two-sided stable minimal surface with intrinsic curvature at least $\frac{S}{2}$. Then

$$(|B|^2)^{(n)}(0, x) = 0.$$

The Claim of Cai and Galloway referred to above is the following:

(M, g) be a Riemannian 3-manifold with non-negative scalar curvature. Suppose that Σ is an embedded stable minimal torus. If η is a normal vector field on Σ , we have

$$(|B|^2)^{(n)}(0, x) = 0.$$

We refer the reader to [3] for the complete proof. Here we make a few comments about it. The original proof is by induction on the vanishing of the n th derivatives of $\lambda_i(0, x)$, $S(0, x)$ and $K(0, x)$, where $\lambda_i(t, x)$ are the eigenvalues of B_t , $S(t, x)$ the scalar curvature of M and $K(t, x)$ the intrinsic curvature of Σ_t . An inspection of the proof shows that the arguments go through if we have constant scalar curvature S and $K(0, x) = \frac{1}{2}S(0, x)$.

Let us check that $K(0, x) = \frac{1}{2}S(0, x)$, in our case. We know that the scalar curvature of (M, g) is $S = 6$. Our assumption in the theorem we are trying to prove is that $K(0, x) \leq 3$. To see that $K(0, x) \equiv 3$, we note that the stability of Σ implies the second variation formula for the area functional, i.e.,

$$\int_{\Sigma} (f^2 |B|^2 + f^2 \text{Ric}_g(v, v) - |\nabla f|^2) dA \leq 0$$

for any f in $C^1(\Sigma)$. Letting $f = 1$, we get

$$\int_{\Sigma} \text{Ric}_g(v, v) \leq 0.$$

But $\text{Ric}_g(v, v) = 3 - K \geq 0$. Hence, we get $K \equiv 3$ on Σ .

By Corollary 2.4, g is analytic. Hence the result above will imply that $B_t = 0$ for all t . Hence, by (18), we find that the area is constant along the variation. So we have an locally area minimizing surface in the interior and we can refer to Case I.

To complete the proof of the theorem, we find the structure of N .

In both Cases I and II, we get, up to isometry,

$$(M, g) = (I \times S^2, dt^2 + g_{S^2(a)}),$$

where $I = [c, d]$ is some interval and a as above. Note that $\text{Ric}_g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$, $\text{Ric}_g(X, X) = 3$ and $\text{Ric}_g(X, \frac{\partial}{\partial t}) = 0$, where X is any tangent vector to the S^2 factor.

Hence, (1) gives $D^2u(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -3u$, $D^2u(X, X) = 0$ and $D^2u(X, \frac{\partial}{\partial t}) = 0$.

The second equation implies that $\Delta u = 0$ on $\{t\} \times S^2$, for each t . Hence, u is constant on $\{t\} \times S^2$, i.e., $u(t, x) = u(t)$.

The first equation gives

$$u'' = -3u.$$

By Corollary 2.4, u should satisfy the boundary conditions $u = 0$ and $|du| = 1$ on ∂M . Hence $u(c) = u(d) = 0$ and $u'(c) = 1$, $u'(d) = -1$. Solving, we get $u = a \sin(\frac{t}{a})$ and $I = [0, \pi]$.

Therefore

$$N = \frac{[0, \pi] \times S^2 \times S^1}{\sim}, \quad h = dt^2 + g_{S^2(a)} + a^2 \sin^2\left(\frac{t}{a}\right) d\theta^2,$$

(notation as in the Introduction), which is isometric to $(S^2 \times S^2, g_{S^2(\frac{1}{\sqrt{3}})} \times g_{S^2(\frac{1}{\sqrt{3}})})$. $\square$

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