HOLOMORPHIC MOTIONS AND
COMPLEX GEOMETRY

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Abstract. We show that the graph of a holomorphic motion of the unit disc cannot be biholomorphic to a strongly pseudoconvex domain in $\mathbb{C}^n$.

1. Introduction and Main Result

Let $B$ be a connected complex $(n-1)$-manifold with a basepoint $z_0 \in B$ and let $\Delta \subset \mathbb{C}$ denote the unit disc. A holomorphic motion of $\Delta$ parametrized by $B$ is a map $f : B \times \Delta \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ satisfying the following conditions:

1. $f(z_0, w) = w$ for all $w \in \Delta$,
2. $f(z, ) : \Delta \to \mathbb{CP}^1$ is injective for each $z \in B$,
3. $f(, w) : B \to \mathbb{CP}^1$ is holomorphic for each $w \in \Delta$.

Holomorphic motions were introduced by R. Māne, P. Sad and D. Sullivan [12] and have been intensively studied since then (see, for instance, [14, 6, 1, 5]). In this note we study the complex-analytic structure of the graph $D$ of $f$:

$$D = \{(z, f(z, w)), z \in B, w \in \Delta\} \subset B \times \mathbb{CP}^1.$$

Our main result is the following

Theorem 1.1. The graph $D$ of a holomorphic motion of the unit disc cannot be biholomorphic to a strongly pseudoconvex domain in $\mathbb{C}^n$.

By a $n$-ball hyperplane we will mean a non-empty intersection of a complex affine hyperplane in $\mathbb{C}^n$ with $\mathbb{B}^n$, the unit ball in $\mathbb{C}^n$. We say that a set $\mathcal{F}$ of $n$-ball hyperplanes fills $\mathbb{B}^n$ if every $p \in \mathbb{B}^n$ lies in some $F_p \in \mathcal{F}$.

Theorem 1.1 will be a consequence of the following result.

Proposition 1.2. Let $\mathcal{F}$ be a set of $n$-ball hyperplanes which fills $\mathbb{B}^n$. There is no holomorphic map $\pi : \mathbb{B}^n \to \mathbb{B}^{n-1}$ such that $\pi|_F : F \to \mathbb{B}^{n-1}$ is bijective for every $F \in \mathcal{F}$.

To derive Theorem 1.1 from Proposition 1.2, we use a rescaling argument based on a recent result of K. T. Kim and L. Zhang [9].

To put these results in context, we recall the following result of K. Liu [11] and V. Koziarz-N. Mok [10]:

Theorem 1.3. ([11], [10]) Let $n > m \geq 1$ and let $\Gamma_1 \subset SU(n, 1)$, $\Gamma_2 \subset SU(m, 1)$ be torsion-free cocompact lattices. Then there does not exist a holomorphic submersion from $\mathbb{B}^n/\Gamma_1$ to $\mathbb{B}^m/\Gamma_2$.

This was proved for $n = 2, m = 1$ by K. Liu [11] and for all $n > m \geq 1$ by V. Koziarz and N. Mok [10]. This result is natural from various points of view. In particular, it is related to the following well-known question in the study of negatively curved Riemannian manifolds:

Let $f : M^n \to N^m$ be a smooth fibre bundle where $M$ and $N$ are smooth compact manifolds of dimensions $n > m \geq 2$. Can $M$ admit a Riemannian metric with negative sectional curvature?
Proposition 1.2 implies the Liu/Koziarz-Mok result when \( n = m + 1 \) by the Bers-Griffiths uniformization theorem as explained later in the paper. The compactness of the manifolds is essential in the question above and the result of Liu-Koziarz-Mok. In other words, cocompact discrete group actions on universal covers are needed. Our point of view is that given the Bers-Griffiths theorem (Theorem 2.6 below), cocompact discrete group actions are not necessary. The proof we present involves some elementary facts about the Kobayashi metric and Riemannian geometric techniques.

It follows that Theorem 1.1 that \( \mathbb{B}^n \) cannot be biholomorphic to the graph of a holomorphic motion. In other words, the graph cannot admit a complete Kähler metric with constant negative holomorphic sectional curvature. Hence the following question is natural:

*Can the graph of a holomorphic motion of the unit disc admit a complete Kähler metric with variable negative sectional curvature?*

A related question, mentioned to the authors by Benoit Claudon and Pierre Py, is:

*Can the graph of a holomorphic motion of the unit disc be Gromov hyperbolic with respect to the Kobayashi metric?*

The method in this paper appears to hold some promise for tackling these questions. In this connection, it is important to point out that metrics with weaker negative curvature conditions can exist on such domains: a result of S. K. Yeung [16] asserts that the universal cover of a Kodaira fibration, which is necessarily the graph of a holomorphic motion by the Bers-Griffiths theorem, admits complete Kähler metrics with negative holomorphic bisectional curvature.

While Theorem 1.1 may appear obvious at first sight, the lack of holomorphicity of \( f \) in the second variable is a nontrivial obstruction to proving it. Indeed, it is natural to attempt a proof by adapting classical methods from multivariable complex analysis, for instance the ones involved in the proof of the non-equivalence between the unit ball and the polydisk in \( \mathbb{C}^n \). However, to the best of our knowledge, these methods do not yield the desired result.

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## 2. Proof of Proposition 1.2

We assume that there are maps \( f \) and \( \pi \) as in Proposition 1.2 and obtain a contradiction.

If \( (X, d_X) \) and \( (Y, d_Y) \) are metric spaces and \( f : X \to Y \) is an isometric embedding we will refer to \( f \) as a *totally geodesic* embedding and we will say that the image \( f(X) \) is *totally geodesic* in \( Y \).

We will need the following standard facts:

(i) The Kobayashi metric on \( \mathbb{B}^n \) is the norm of the Bergman metric, which is also the complex hyperbolic metric. For the rest of the paper, \( \mathbb{B}^n \) will always be endowed with the complex hyperbolic metric.

(ii) The totally geodesic complex submanifolds of \( \mathbb{B}^n \) are precisely the intersections of complex affine subspaces of \( \mathbb{C}^n \) with \( \mathbb{B}^n \).

### 2.1. The fibers of \( \pi \).

In what follows we use the following notation: for any \( p \in \mathbb{B}^n \),

\[ S_p := \pi^{-1}(\pi(p)). \]
Lemma 2.1. Let \( p \in \mathbb{B}^n \) and let \( p \in F_p \) for some \( F_p \in \mathcal{F} \). If \( \gamma : [0, \infty) \rightarrow \mathbb{B}^n \) is a geodesic with \( \gamma(0) = p \) satisfying one of the following two conditions: either \( \gamma'(0) \in T_p F_p \) or \( \gamma(L) \in F_p \) for some \( L > 0 \), then

(i) \( \gamma([0, \infty)) \subset F_p \) and

(ii) \( \gamma|_{[0, s]} \) is a shortest geodesic between \( S_p \) and \( S_{\gamma(s)} \) for any \( s > 0 \).

**Proof:** Since \( F_p \) is an intersection of a complex hyperplane with \( \mathbb{B}^n \), it is a totally geodesic submanifold. Hence \( \gamma'(0) \in T_p F_p \) implies that \( \gamma([0, \infty)) \subset F_p \). If \( \gamma(L) \in F_p \), then the uniqueness of geodesics between two points (which follows from the strict negative curvature of \( \mathbb{B}^n \)) again implies \( \gamma([0, \infty)) \subset F_p \).

Let \( s > 0 \), \( p' \in S_p \) and \( q \in S_{\gamma(s)} \). We have

\[
d^K_{\mathbb{B}^n}(p', q) \geq d^K_{\mathbb{B}^n-1}(\pi(p), \pi(\gamma(s))) = l(\gamma|_{[0, s]}) = d^K_p(p, \gamma(s)),
\]

where \( l(\gamma|_{[0, s]}) = s \) denotes the length of \( \gamma|_{[0, s]} \). The first equality above comes from the assumption that \( \pi|_{F_p} : F_p \rightarrow \mathbb{B}^{n-1} \) is bijective and hence a biholomorphism and an isometry.

Lemma 2.2. (1) For every \( p \in \mathbb{B}^n \) and \( F \in \mathcal{F} \), \( S_p \cap F \) consists of a single point and the intersection is orthogonal.

(2) The fibers of \( \pi \) are equidistant, i.e., for any \( p, q \in \mathbb{B}^n \), we have

\[
d^K_{\mathbb{B}^n}(p, S_q) = d^K_{\mathbb{B}^n}(S_p, S_q).
\]

**Proof:** It is clear that for every \( p \in \mathbb{B}^n \) and \( F \in \mathcal{F} \), the intersection \( S_p \cap F \) consists of a singleton since \( \pi|_{F} : F \rightarrow \mathbb{B}^{n-1} \) is bijective.

Let \( v \in T_p F_p \) and let \( \gamma : [0, \infty) \rightarrow B^n \) be the geodesic satisfying \( \gamma(0) = p \) and \( \gamma'(0) = v \). By Lemma 2.7, \( \gamma|_{[0, s]} \) is a shortest geodesic between \( S_p \) and \( S_{\gamma(s)} \) for any \( s > 0 \). The first variation for arc-length implies that \( \gamma \) meets \( S_p \) and \( S_{\gamma(s)} \) orthogonally. In particular, \( v = \gamma'(0) \) is orthogonal to \( T_p S_p \). This proves (1).

If \( p, q \in \mathbb{B}^n \), then \( F_p \cap S_q \) consists of a single point, say \( q' \), by (1). Let \( \gamma : [0, L] \rightarrow \mathbb{B}^n \) be a geodesic with \( \gamma(0) = p \) and \( \gamma(L) = q' \). By Lemma 2.7, \( \gamma|_{[0, L]} \) is a shortest geodesic between \( S_p \) and \( S_q \), proving (2).

**Remark:** Even though we will not use this fact, it follows from Lemma 2.2 Part (1) that if \( F_1, F_2 \in \mathcal{F} \), then either \( F_1 = F_2 \) or \( F_1 \cap F_2 = \emptyset \). This is seen as follows: if \( p \in F_1 \cap F_2 \), then \( T_p F_1 = (T_p S_p)^\perp = T_p F_2 \). Since \( F_1 \) and \( F_2 \) are closed, totally geodesic submanifolds, equality of tangent spaces at an intersection point implies that they are equal.

**Corollary 2.3.** Let \( \gamma : [0, \infty) \rightarrow \mathbb{B}^n \) be a unit speed geodesic with \( \gamma(0) = p \), \( \gamma'(0) \in (T_p S_p)^\perp \). If \( P_s \) denotes the parallel transport of \( T_p S_p \) along \( \gamma \), then

\[
P_s = T_{\gamma(s)} S_{\gamma(s)}.
\]

**Proof:** (i) Let \( \{e_1, e_2, \ldots, e_{2n}\} \) be an orthonormal basis of \( T_p \mathbb{B}^n \) with \( e_1, e_2 \in T_p S_p \) and let \( E_i(s) \) be the parallel translate of \( e_i \), \( i = 1, \ldots, 2n \), along \( \gamma \). As before, \( \gamma \) lies in \( F_p \). By Lemma 2.2 Part (1), \( e_i \in T_p F_p \) for \( 3 \leq i \leq 2n \). Since \( F_p \) is totally geodesic, \( E_i(s) \) is tangent to \( F_p \) for all \( s \in [0, L] \) and \( 3 \leq i \leq 2n \). Hence \( E_1(s), E_2(s) \in (T_{\gamma(s)} S_p)^\perp = T_{\gamma(s)} S_{\gamma(s)} \).

2.2. Submanifold geometry. We recall some basic constructions in submanifold geometry.

- Let \((N, g)\) be a Riemannian manifold and \( M \subset N \) be a submanifold. For \( p \in M \) and \( \nu \in (T_p M)^\perp \) the shape operator \( L_\nu \) of \( M \) at \( p \) along \( \nu \) is the linear operator \( L_\nu : T_p M \rightarrow T_p M \) defined for any \( u \in T_p M \) by

\[
L_\nu(u) = (\nabla_u \nu)^T.
\]
Here \( \tilde{\nu} \) is a local vector field normal to \( M \), with \( \tilde{\nu}_p = \nu \), and if \( v \in T_pN \), \( v^T = v - v^\perp \in T_pM \) denotes the tangential component of \( v \).

It is easy to check that

(i) \( L_\nu \) depends only on \( \nu \) and not on the choice of an extension \( \tilde{\nu} \),

(ii) \( L_\nu \) is a symmetric operator with respect to the Riemannian inner product on \( T_pM \) induced from \( g \).

- The second fundamental form \( B_\nu \) of \( M \) at \( p \) along \( \nu \) is the symmetric bilinear form on \( T_pM \) associated to \( L_\nu \), i.e.,

\[
B_\nu(u, v) = \langle u, L_\nu(v) \rangle = \langle u, \nabla_\nu \tilde{\nu} \rangle,
\]

for any \( u, v \in T_pM \) and the normal vector field \( \tilde{\nu} \) is as before.

The mean curvature \( H_\nu \) of \( M \) at \( p \) along \( \nu \) is

\[
H_\nu = \text{trace } B_\nu = \sum_{i=1}^{k} B_\nu(e_i, e_i),
\]

where \( \{e_i\}_{i=1}^{k} \) is any orthonormal basis of \( T_pM \).

- Now suppose that \( (N, g) \) is a Kähler manifold, \( M \) is a complex submanifold of \( N \) and \( \xi \in (T_pM)^\perp \). Then

\[
L_{J\xi} = JL_\xi = -L_\xi J,
\]

where \( J : T_pN \to T_pN \) denotes the almost complex structure.

Moreover, the mean curvature of \( M \) along any normal vector is zero.

2.3. The second fundamental forms of fibers of \( \pi \). Let \( p \in \mathbb{B}^n \). We are interested in the shape operator of fiber \( S_p \) along a unit vector \( \nu \in (T_pS_p)^\perp \). We begin by choosing a specific extension \( X \) of \( \nu \) as follows. By (2) of Lemma 2.2 and the implicit function theorem applied to the normal exponential map to \( S_p \), we can find a unit normal vector field \( X \) in a neighbourhood \( U \) (in \( S_p \)) of \( p \) with the following properties:

(i) \( X_p = \nu \),

(ii) there exists \( r > 0 \) such that for any \( q \in U \), \( \exp_q(rx) \in S_{\exp_p(rv)} \). We remark that the proof of (i) of Lemma 2.4 below shows that \( \exp_q(sX_q) \in S_{\exp_p(sv)} \) for all \( 0 < s \leq r \).

Let \( \gamma(s) = \exp_p(sv) \), \( s \in [0, r] \). We calculate the eigenvalues of the shape operator of \( S_{\gamma(s)} \) at \( \gamma(s) \) along the normal \( \gamma'(s) \). We do this by using Jacobi fields, as below.

Let \( u \in T_pS_p \) and \( \sigma : [-a, a] \to S_p \) a curve with \( \sigma(0) = p \) and \( \frac{d\sigma}{dt}|_{t=0} = u \). Define a geodesic variation \( H^u : [0, \infty) \times [-a, a] \to \mathbb{B}^n \) of \( \gamma \) by

\[
(2.2) \quad H^u(s, t) = \exp_{\sigma(t)}(sX_{\sigma(t)}).
\]

Note that the variation above (and the associated quantities below) actually depends on \( \sigma \) and not just \( u = \frac{d\sigma}{dt}|_{t=0} \).

Let

\[
Y^u(s, t) = \frac{\partial H^u}{\partial t}(s, t), \quad T^u(s, t) = \gamma'_t(s) = \frac{\partial H^u}{\partial s}(s, t)
\]

be the variation vector fields.

Note that

\[
T^u(0, t) = X_{H^u(0, t)}, \quad Y^u(0, 0) = u.
\]

For each \( t \in [-a, a] \), let \( \gamma_t \) be the geodesic given by

\[
\gamma_t(s) = H^u(s, t).
\]
We have 
\[ \gamma'_1(s) = T^u(s, t) \] and \( \gamma_0 = \gamma \).

**Lemma 2.4.** For any \((s, t) \in [0, r] \times [-a, a]\), we have

(i) \( Y^u(s, t) \in T_{\gamma_1(s)}S_{\gamma_1(s)} \).

(ii) \((Y^u)'(s, t) := \nabla_T Y^u(s, t) \in T_{\gamma_1(s)}S_{\gamma_1(s)} \).

(iii) \( \nabla_u X(p) \in T_pS_p \).

**Proof:** (i) This follows if we can show that for each \( s \in [0, r] \) the curve \( t \mapsto H^u(s, t) \) lies in a fiber of \( \pi \). To see, let \( t \in (-a, a) \) and consider the curve \( s \mapsto (\pi \circ \gamma_t)(s) \). By assumption, \( \gamma_t(L) \in S_{\gamma_t(L)} \) for some \( L > 0 \), i.e., \( \pi \circ \gamma_t(L) = \pi \circ \gamma(L) \). Since \( \gamma_t \) is a geodesic lying in some \( F \in F \) and \( \pi|_F \) is an isometry, \( \pi \circ \gamma_t \) and \( \pi \circ \gamma \) are unit-speed geodesics in \( \mathbb{B}^{n-1} \) connecting \( \pi(p) \) and \( \pi \circ \gamma(L) \). As \( \mathbb{B}^{n-1} \) has negative curvature, uniqueness of geodesics forces \( \pi \circ \gamma_t(s) = \pi \circ \gamma(s) \) for all \( s \in [0, r] \).

(ii) We show this for \( t = 0 \) for notational simplicity. Let \( \{e_1, e_2\} \) be an orthonormal basis of \( T_{p_0}S_{p_0} \). Let \( E_1(s), E_2(s) \) be the parallel vector fields along \( \gamma \) with \( E_i(0) = e_i \). We then have, by (i) and Corollary 2.3,

\[ Y(s, 0) = f_1(s)E_1(s) + f_2(s)E_2(s) \]

for some functions \( f_1, f_2 : [0, r] \to \mathbb{R} \). Hence

\[ Y'(s, 0) = f_1'(s)E_1(s) + f_2'(s)E_2(s) \in T_{\gamma_1(s)}S_{\gamma_1(s)} \).

(iii) Since \( [Y^u, T^u] = 0 \), (ii) above implies that \( \nabla_u X = \nabla_{Y^u(s, 0)}Y^u \in T_pS_p \).

\[ \square \]

**Lemma 2.5.** Let \( B^s \) denote the second fundamental form of \( S_{\gamma(s)} \) at \( \gamma(s) \) along the normal direction \( \gamma'_1(s) \). Then

\[ B^s(Y^u(s, 0), Y^u(s, 0)) = \frac{1}{2}(|Y^u|^2)'(s, 0). \]

**Proof:**

\[ B^s(Y^u(s, 0), Y^u(s, 0)) = \langle Y^u, \nabla_{Y^u}T^u \rangle(s, 0) = \langle Y^u, \nabla_{T^u}Y^u \rangle(s, 0) = \frac{1}{2}(|Y^u|^2)'(s, 0) \]

where the second equality follows from \( \nabla_{Y^u}T^u = \nabla_{T^u}Y^u \).

\[ \square \]

Let \( L^s \) denote the shape operator of \( S_{\gamma(s)} \) at \( \gamma(s) \) along the normal direction \( \gamma'_1(s) \) and let \( \{e_1, e_2\} \) be an orthonormal basis of \( T_pS_p \) consisting of eigenvectors of \( L^0 \). Since \( S_p \) is a minimal submanifold (being a complex subvariety), we can assume that the corresponding eigenvalues are of the form \( \alpha, -\alpha \) for some \( \alpha \geq 0 \). As before we denote the parallel transports of \( e_1, e_2 \) along \( \gamma \) by \( E_1(s), E_2(s) \).

Let \( Y_i(s) := Y^e_i(s, 0), i = 1, 2 \). Then

\[ Y_i(0) = e_i, \quad i = 1, 2, \quad Y'_1(0) = \alpha e_1, \quad Y'_2(0) = -\alpha e_2, \]

since \( Y'_i(0) = \nabla_{Y_i(0)}Y_i = \nabla_{e_i}X = L^0(e_i) \) by (iii) of Lemma 2.4.

Next, we recall the following elementary facts:

(i) if \( Y \) is a Jacobi field along a geodesic \( \gamma \), then \( Y \) satisfies the ordinary differential equation

\[ Y''(s) + R(Y(s), \gamma'(s))\gamma'(s) = 0. \]

(ii) the curvature tensor \( R \) of complex hyperbolic space is given by

\[ 4(R(X, Y), Z, W) = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \langle X, JZ \rangle \langle Y, JW \rangle - \langle X, JW \rangle \langle Y, JZ \rangle + 2\langle X, JY \rangle \langle Z, JW \rangle \]

\[ + 2\langle X, JZ \rangle \langle Y, JW \rangle - 2\langle X, JW \rangle \langle Y, JZ \rangle. \]
Since \( \langle E_1(s), \gamma'(s) \rangle = \langle E_1(s), J\gamma'(s) \rangle = 0 \), (2.3) implies that
(i) \( R(E_1(s), \gamma'(s))\gamma'(s), X \) = 0 for any \( X \) with \( \langle E_1(s), X \rangle = 0 \).
(ii) \( R(E_1(s), \gamma'(s))\gamma'(s), E_1(s) \) = \(-\frac{1}{4}\).

Hence if \( f_1 \) and \( f_2 \) are the solutions to
\[
y'' - \frac{y}{4} = 0
\]
satisfying the initial conditions \( f_1(0) = f_2(0) = 1 \) and \( f_1'(0) = -f_2'(0) = \alpha \), then
\[
Y_1 = f_1E_1, \quad Y_2 = f_2E_2.
\]

Solving, we find:
\[
f_1(s) = \cosh(\frac{s}{2}) + 2\alpha \sinh(\frac{s}{2}), \quad f_2(s) = \cosh(\frac{s}{2}) - 2\alpha \sinh(\frac{s}{2}).
\]

2.4. **Proof of Proposition 1.2.** Let \( \alpha \) be as in the previous section.

**Case 1:** \( \alpha \neq \frac{1}{2} \).

Lemma 2.5 implies that
\[
B^s(Y_i(s, 0), Y_i(s, 0)) = \frac{1}{2}(\langle [Y_i]^2 \rangle)(s, 0) = \frac{1}{2}(f_i^2)'(s)
\]
for \( i = 1, 2 \). On the other hand, the fact that \( S_{\gamma(s)} \) is a minimal submanifold implies that
\[
0 = \sum_{i=1}^{2} B^s(E_i(s), E_i(s)) = \sum_{i=1}^{2} f_i^{-2} B^s(Y_i(s, 0), Y_i(s, 0)) = \sum_{i=1}^{2} \frac{f_i'}{f_i} = ((\log(f_1f_2))')
\]
for all \( s \in [0, r_0] \) where \( r_0 \leq r \) is small enough that \( f_i(s) \neq 0 \) for all \( s \in [0, r_0] \).

Hence we conclude that \( f_1f_2 \) is a constant which is possible only if \( \alpha = \frac{1}{2} \) (recall that we had assumed \( \alpha \geq 0 \)). This contradiction completes the proof.

**Case 2:** \( \alpha = \frac{1}{2} \) for all \( p \in \mathbb{B}^n \), all \( q \in S_p \) and all \( n \in (T_qS_p)^\perp \).

**Claim:** Let \( S \) be a fiber of \( \pi \). The sectional curvature of the induced metric on \( S \) is zero at any \( q \in S \).

This will contradict the fact that the sectional curvatures of a complex submanifold of a Kähler manifold with strictly negative sectional curvature are strictly negative.

We change notations slightly and denote the Riemannian connection of \( S \) by \( \nabla \) and that of \( \mathbb{B}^n \) by \( \tilde{\nabla} \).

Let \( q \in S \). Choose an orthonormal basis \( \{\xi_1, \ldots, \xi_{2n-2}\} \) of \( (T_qS)^\perp \) of the form \( \xi_{2k} = J\xi_{2k-1} \) for \( k = 1, \ldots, n-1 \). Extend \( \xi_{2k-1}, \ k = 1, \ldots, n-1, \) to unit normal vector fields \( X_{2k-1} \) (defined on a neighborhood \( U \) of \( q \)) satisfying \((i)\), in the sense that \( X_{2k-1}(p) = \xi_{2k-1}, \) and \((ii)\) in Page 4. Observe that for such vector fields, \((iii)\) of Lemma 2.4 implies
\[
\langle \tilde{\nabla}_uX_k(x) \rangle^\perp = 0
\]
for \( k = 1, 3, \ldots, 2n-3 \) and all \( x \in U \) and all \( u \in T_xS \). Let \( X_{2k} = JX_{2k-1} \) for \( k = 1, \ldots, n-1 \).

Since \( \langle \tilde{\nabla}_uJX_{2k-1} \rangle^\perp = J\langle \tilde{\nabla}_uX_{2k-1} \rangle^\perp \), we can assume that (2.4) holds for all \( k \).

Equation (2.4) implies that \( \{X_1, \ldots, X_{2n-2}\} \) is an orthonormal frame for the normal bundle of \( S \) on \( U \) since, for any \( 1 \leq j, k \leq 2n-2, \ x \in U \) and \( u \in T_xS \), we have
\[
u(\langle X_j, X_k \rangle) = \langle \tilde{\nabla}_uX_j, X_k \rangle + \langle X_j, \tilde{\nabla}_uX_k \rangle = 0
\]
by (2.4). Therefore \( \langle X_j(x), X_k(x) \rangle = \langle X_j(q), X_k(q) \rangle = 0 \) if \( j \neq k \).
Consider the shape operators \( L_k := L_{X_k} \). The hypothesis on eigenvalues of shape operators gives
\[
4L_k^2 = I,
\]
where \( I \) is the identity operator.

We claim that
\[
\nabla_u L_k = 0
\]
for any \( x \in U, u \in T_x S \) and \( 1 \leq k \leq 2n - 2 \). We check this for \( k = 1 \), the general case is similar: take the covariant derivative, along \( u \), of (2.5) to get
\[
(\nabla_u L_1) L_1 + L_1 (\nabla_u L_1) = 0.
\]
Since \( L_1 \) is symmetric and trace-free (as \( S \) is a complex submanifold), so is \( \nabla_u L_1 \). Hence, the above equation implies that
\[
\nabla_u L_1 = \phi_1(u) L_2 = \phi_1 J L_1
\]
for some smooth locally defined 1-form \( \phi_1 \) on \( S \). Here we have used \( L_2 = J L_1 = -L_1 J \).

Next, let \( s_{jk} \), with \( j, k = 1, \ldots, 2n - 2 \), denote the normal connection 1-forms for the frame \( \{X_1, \ldots, X_{2n-2}\} \), defined by (see (7.4), Page 388 of [3])
\[
(\tilde{\nabla}_u X_j) = \sum_k s_{jk}(u) X_k.
\]
In our setting, (2.4) implies
\[
s_{jk} = 0, \quad j, k = 1, \ldots, 2n - 2
\]
on \( U \).

The Codazzi equations for a complex submanifold in a complex space-form imply that (see (7.20), Page 392 of [3])
\[
(\nabla_u L_1)(v) - (\nabla_v L_1)(u) = \sum_k (s_{1k}(u)L_k(v) - s_{1k}(v)L_k(u)).
\]
Using (2.7) in the above equation and invoking (2.8) gives \( \phi_1 \equiv 0 \), i.e., \( \nabla_u L_1 = 0 \) on \( U \). The same argument gives \( \nabla_u L_k = 0 \) for all \( k \).

Since the eigenvalues of \( L_1 \) are distinct, we can find a smooth local frame \( \{e_1, J e_1\} \) (for the tangent bundle of \( S \)) near \( q \) consisting of unit-norm eigenvectors of \( L_1 \). Taking the covariant derivative of \( L_1(e_1) = \frac{1}{2} e_1 \) and using (2.6), we get
\[
L_1(\nabla_u e_1) = \frac{1}{2} \nabla_u e_1.
\]
Hence \( \nabla_u e_1 = \phi(u) e_1 \) for some smooth local 1-form \( \phi \) on \( S \). Since \( e_1 \) has unit norm, we have \( 0 = u(\langle e_1, e_1 \rangle) = 2 \langle \nabla_u e_1, e_1 \rangle = 2\phi(u) \). Hence \( \nabla_u e_1 = 0 \), for any \( u \in T_x S \) for \( x \) close to \( q \). This implies \( \nabla_u J e_1 = 0 \), as well. We conclude that \( S \) has zero sectional curvature near \( q \).

2.5. **Holomorphic motions and Bers-Griffiths uniformization.** In this section we describe how Theorem 1.3 follows from Theorem 1.2 when \( n = m + 1 \).

The starting point is the following fundamental result:

**Theorem 2.6.** ([2], [7]) Let \( M \) and \( N \) be compact complex manifolds and \( \phi : M \to N \) a holomorphic submersion. Suppose that \( \dim(M) = \dim(N) + 1 \) and the fibers of \( \phi \) are compact Riemann surfaces of genus \( \geq 2 \). Then the universal cover of \( M \) is biholomorphic to the graph of a holomorphic motion over the universal cover of \( N \).
For a detailed account of holomorphic motions and uniformization we refer the reader to [5].

Combining this with the following observation, we get a \( n \)-ball hyperplane filling as in Proposition 1.2:

Let \( B \) be a connected complex \((n - 1)\)-manifold with a basepoint \( z_0 \in B \); \( f : B \times \Delta \rightarrow \mathbb{CP}^1 \) a holomorphic motion, \( D := \{(z, f(z, w)) : z \in B, w \in \Delta \} \subset B \times \Delta \) and \( \pi : D \rightarrow B \) the first projection.

Proposition 2.7. For every \( w \in \Delta \), the map \( F_w : B \times \{w\} \rightarrow D \) given by \( F_w(z) = (z, f(z, w)) \) is a totally geodesic holomorphic embedding for the Kobayashi metrics on \( B \) and \( D \).

Proof: For every \( z_1, z_2 \in B \) and every \( w \in \Delta \),

\[
d_B^K(z_1, z_2) \geq d_B^K(F_w(z_1), F_w(z_2)) \geq d_\Delta^K(\pi \circ F_w(z_1), \pi \circ F_w(z_2)) = d_B^K(z_1, z_2)
\]

by the distance decreasing property of the Kobayashi metric. This proves the result. \( \square \)

Note that \( \cup_{w \in \Delta} F_w(\Delta) = D \). In particular, if \( D \) is biholomorphic to \( \mathbb{B}^n \) then the hypotheses of Proposition 1.2 are satisfied.

3. Proof of Theorem 1.1.

We keep the same notation as in Section 1. The main result of this section is:

Proposition 3.1. Suppose that a graph \( D \) of a holomorphic motion of a disc over a connected complex manifold \( B^{n-1} \) is biholomorphic to a bounded strictly pseudoconvex domain \( \Omega \subset \mathbb{C}^n \). Then there exists an \( n \)-ball hyperplane filling \( F \) and a holomorphic map \( \pi : \mathbb{B}^n \rightarrow \mathbb{B}^{n-1} \) such that \( \pi|_F : F \rightarrow \mathbb{B}^{n-1} \) is bijective for every \( F \in \mathcal{F} \).

Since Proposition 1.2 asserts that such a filling cannot exist, this will prove Theorem 1.1.

Throughout this section we use the following notation:

- \( \Phi \) will denote a biholomorphism from \( D \) to \( \Omega \).
- For every \( k \geq 1 \), \( F_k : B \rightarrow D \) is the totally geodesic holomorphic embedding defined by

\[
F_k(z) := F(z, 1 - \frac{1}{k})
\]

where \( F(z, w) = (z, f(z, w)) \) for every \((z, w) \in B \times \Delta \).

- Let \( z_0 \in B \). Then \((z^k := \Phi \circ F_k(z_0))_k \) is a sequence of points in \( \Omega_k \) such that

\[
\lim_{k \rightarrow \infty} z^k = p \in \partial \Omega.
\]

According to [9, Theorem 4.1], \( \lim_{z \rightarrow p} \sigma_{\Omega}(z) = 1 \), where \( \sigma_{\Omega} \) is the squeezing function of \( \Omega \) (see Definition in [9]). This means the following:

- For every \( k \geq 1 \), there exists a biholomorphism \( \varphi_k \) from \( \Omega \) to some strongly pseudoconvex domain \( \Omega_k \) and there exists \( 0 < r_k < 1 \), with \( \lim_{k \rightarrow \infty} r_k = 1 \), such that:

\[
\varphi_k(z^k) = 0 \text{ and } B(0, r_k) \subset \Omega_k \subset \mathbb{B}^n.
\]

Here \( B(0, r_k) \) denotes the ball in \( \mathbb{C}^n \) centered at the origin with radius \( r_k \).
- For every \( k \geq 1 \), \( \Psi_k : B \rightarrow \Omega_k \) is the holomorphic isometric embedding defined by

\[
\Psi_k := \varphi_k \circ \Phi \circ F_k.
\]
- For every \( k \geq 1 \), let

\[
\Sigma_k := F_k(B) \subset D, \quad \tilde{\Sigma}_k := (\varphi_k \circ \Phi)(\Sigma_k) = \Psi_k(B) \subset \Omega_k.
\]
Lemma 3.2. For every \( k \geq 1 \), the set \( \tilde{\Sigma}_k^k \) is a totally geodesic complex submanifold of \( \Omega_k \).

Proof: This follows from Lemma 2.7 and the fact that biholomorphisms are isometries for the Kobayashi metric.

Moreover, we get:

Lemma 3.3. The sequence \( (\tilde{\Sigma}_k^k) \) converges, for the local Hausdorff convergence of sets, to some totally geodesic complex submanifold of \( \mathbb{B}^n \).

Proof: Note that the holomorphic isometric embedding of \((B, d^K_B)\) into \((\Omega_k, d^K_{\Omega_k})\) given by \( \Psi_k := \varphi_k \circ \Phi \circ F_k \) satisfies \( \Psi_k(z_0) = 0 \). Since for every \( k \geq 1 \) we have the inclusion \( \Omega_k \subset \mathbb{B}^n \), the sequence \( (\Psi_k)_k \) is normal and extracting a subsequence if necessary, we may assume that \((\Psi_k)_k\) converges, uniformly on compact subsets of \( B \), to some holomorphic map \( \Psi_\infty : B \to \mathbb{B}^n \) satisfying \( \Psi_\infty(z_0) = 0 \). Finally, let \( 0 \in L \subset \subset \mathbb{B}^n \). Since \( \Psi_k \) is an isometry for the Kobayashi distances, there exists \( K \subset \subset B \) such that for every \( k \geq 1 \) we get: \( L \cap \tilde{\Sigma}_0^k \subset \Psi_k(K) \). Now the uniform convergence of \((\Psi_k)_k\) on \( K \) implies that the sets \( \tilde{\Sigma}_0^k \) converge to \( \tilde{\Sigma}_0^\infty := \Psi_\infty(B) \) for the Hausdorff convergence on \( L \).

Let \( z, z' \in B \). There exist \( q_k, q'_k \in \tilde{\Sigma}_0^k \), converging respectively to \( \Psi_\infty(z) \) and \( \Psi_\infty(z') \), and we have by Lemma 3.2:

\[
\lim_{k \to \infty} d^K_{\tilde{\Sigma}_0^k}(\Psi_\infty(z), \Psi_\infty(z')) = \lim_{k \to \infty} d^K_{\tilde{\Sigma}_0^k}(\Psi_k(z), \Psi_k(z')) = d^K_{\mathbb{B}^n}(\Psi_\infty(z), \Psi_\infty(z')).
\]

In particular, since totally geodesic complex submanifolds of \( \mathbb{B}^n \), of complex dimension \((n-1)\), are intersections of \( \mathbb{B}^n \) with complex affine subspaces of complex dimension \((n-1)\), we may assume that \( \tilde{\Sigma}_0^\infty = \mathbb{B}^{n-1} \times \{0\} \).

Let \( q \in \mathbb{B}^n \). Then \( q \in \Omega_k \) for sufficiently large \( k \) and there exists \((b_k^q, \zeta_k^q) \in B \times \Delta \) such that \( q = \varphi_k \circ \Phi \circ F_{\zeta_k^q}(b_k^q) \). We set

\[ \tilde{\Sigma}_q^k := \varphi_k \circ \Phi \circ F_{\zeta_k^q}(B). \]

We prove, exactly as in Lemma 3.2 for \( \tilde{\Sigma}_0^k \), that \( \tilde{\Sigma}_q^k \) is a totally geodesic complex submanifold of \( \Omega_k \).

Lemma 3.4. The sequence \((b_q^0)\) is relatively compact in \( B \).

Proof: Since \( D \) is complete hyperbolic by assumption, it follows from Lemma 2.7 that \( B \) is complete hyperbolic. Assume to get a contradiction that \((b_q^0)\) is not relatively compact in \( B \). We recall that \( \pi : D \to B \) is holomorphic. Hence we get for every sufficiently large \( k \):

\[
d^K_D\left(F\left(z_0, 1 - \frac{1}{k}\right), F(b_k, \zeta_k)\right) \geq \lim_{k \to \infty} d^K_B(z_0, b_k^q).
\]

Consequently, extracting a subsequence if necessary, we may assume that:

\[
\lim_{k \to \infty} d^K_D\left(F\left(z_0, 1 - \frac{1}{k}\right), F\left(b_k^q, \zeta_k^q\right)\right) = \infty.
\]

Hence

\[
\lim_{k \to \infty} d^K_{\Omega_k}\left(\varphi_k \circ \Phi\left(F\left(z_0, 1 - \frac{1}{k}\right)\right), \varphi_k \circ \Phi\left(F\left(b_k^q, \zeta_k^q\right)\right)\right) = \infty.
\]

However, since \( \varphi_k \circ \Phi\left(F\left(z_0, 1 - \frac{1}{k}\right)\right) = 0 \) and \( \varphi_k \circ \Phi\left(F\left(b_k^q, \zeta_k^q\right)\right) = q \) for every \( k \) we get:

\[
\lim_{k \to \infty} d^K_{\Omega_k}\left(\varphi_k \circ \Phi\left(F\left(z_0, 1 - \frac{1}{k}\right)\right), \varphi_k \circ \Phi\left(F\left(b_k^q, \zeta_k^q\right)\right)\right) = d^K_{\mathbb{B}^n}(0, q) < \infty.
\]
This contradicts Condition (3.2).

It follows now from Lemma 3.4 that we may extract from \((b_k^n)\) a subsequence, still denoted \((b_k^n)\), that converges to some point \(b_\infty^n \in B\). Hence, extracting a subsequence if necessary, we may assume that the sequence \((\varphi_k \circ \Phi \circ F_{\xi_k})\) converges uniformly (and hence in any \(C^k\) norm) on compact subsets of \(B\) to a holomorphic map \(\Psi_\infty^q : B \to \mathbb{B}^n\) satisfying \(\Psi_\infty^q(b_\infty^n) = q\). This implies that \(\tilde{\Sigma}^k_q = \varphi_k \circ \Phi \circ F_{\xi_k}(B)\) converges to \(\tilde{\Sigma}^\infty_q := \Psi_\infty^q(B)\) and \(\tilde{\Sigma}^\infty_q\) is a totally geodesic complex submanifold of \(\mathbb{B}^n\).

We have finally proved

**Proposition 3.5.** For every \(q \in \mathbb{B}^n\), there exists a totally geodesic complex submanifold \(\tilde{\Sigma}^\infty_q\) of \(\mathbb{B}^n\) passing through \(q\).

For every \(k\), let \(\pi_k : D \to \Sigma_k\) be the holomorphic map

\[
\pi_k(F(z, \zeta)) = F\left(z, 1 - \frac{1}{k}\right)
\]

for any \((z, \zeta) \in B \times \Delta\). Correspondingly, let

\[
\tilde{\pi}_k : \Omega_k \to \tilde{\Sigma}_0^k \text{ be } \tilde{\pi}_k = \varphi_k \circ \Phi \circ \pi_k \circ \Phi^{-1} \circ \varphi_k^{-1}.
\]

Since \(\varphi_k \circ \Phi(F(z_0, 1 - \frac{1}{k})) = 0\) according to (3.1), we have \(\tilde{\pi}_k(0) = 0\) for every \(k\). Hence we may extract from \((\tilde{\pi}_k)_k\) a subsequence, still denoted \((\tilde{\pi}_k)_k\), that converges uniformly on compact subsets of \(\mathbb{B}^n\) to a holomorphic map \(\tilde{\pi}_\infty : \mathbb{B}^n \to \mathbb{B}^{n-1} \times \{0\}\).

Moreover we have:

**Proposition 3.6.** For every \(q \in \mathbb{B}^n\), the restriction of \(\tilde{\pi}_\infty\) to \(\tilde{\Sigma}^\infty_q\) is a biholomorphism from \(\tilde{\Sigma}^\infty_q\) to \(\mathbb{B}^{n-1} \times \{0\}\).

**Proof:** By the very definition of \(\tilde{\pi}_k\), the restriction of \(\tilde{\pi}_k\) to \(\Sigma^k_q\) is a biholomorphism from \(\Sigma^k_q\) to \(\tilde{\Sigma}_0^k\) for every \(k\).

Moreover, \(\tilde{\Sigma}_q^k\) converges to \(\tilde{\Sigma}^\infty_q\) for the Hausdorff distance. Finally, we have for every \(k \geq 1\):

\[
\pi_k \circ \Phi^{-1} \circ \varphi_k^{-1}(q) = \left(b_k^n, 1 - \frac{1}{k}\right) = F_k(b_k^n).
\]

Since \(\lim_{k \to \infty} b_k^n = b^n_\infty \in B\) and since the sequence \((\varphi_k \circ \Phi \circ F_k)_k\) converges, uniformly on compact subsets of \(B\), to \(\Psi_\infty\) (see the proof of Lemma 3.3) we obtain:

\[
\tilde{\pi}_\infty(q) = \lim_{k \to \infty} \varphi_k \circ \Phi \circ \pi_k \circ \Phi^{-1} \circ \varphi_k^{-1}(q) = \Psi_\infty(b_\infty^n) \in \mathbb{B}^{n-1} \times \{0\}.
\]

We prove that the restriction of \(\tilde{\pi}_\infty\) to \(\tilde{\Sigma}^\infty_q\) is one-to-one and onto from \(\tilde{\Sigma}^\infty_q\) to \(\mathbb{B}^{n-1} \times \{0\}\).

Let \(z_\infty, z'_\infty\) be two distinct points in \(\tilde{\Sigma}^\infty_q\). Then there exist, for every \(k\), distinct points \(z_k\) and \(z'_k\) on \(\tilde{\Sigma}^k_q\) such that \(\lim_{k \to \infty} z_k = z_\infty\) and \(\lim_{k \to \infty} z'_k = z'_\infty\). Moreover:

\[
\lim_{k \to \infty} \tilde{\pi}_k(z_k) = \tilde{\pi}_\infty(z_\infty) \text{ and } \lim_{k \to \infty} \tilde{\pi}_k(z'_k) = \tilde{\pi}_\infty(z'_\infty).
\]

The manifolds \(\tilde{\Sigma}^k_q\) and \(\tilde{\Sigma}^\infty_q\) being totally geodesic in \(\Omega_k\) and \(\mathbb{B}^n\) respectively, we get:

\[
d_{\tilde{\Sigma}^k_q}(z_k, z'_k) = d_{\mathbb{B}^n}(z_k, z'_k) \quad \text{and} \quad d_{\tilde{\Sigma}^\infty_q}(z_\infty, z'_\infty) = d_{\mathbb{B}^n}(z_\infty, z'_\infty).
\]

As the domains \(\Omega_k\) converge to \(\mathbb{B}^n\) in the Hausdorff sense on compact subsets of \(\mathbb{B}^n\), we get:

\[
\lim_{k \to \infty} d_{\Omega_k}(z_k, z'_k) = d_{\mathbb{B}^n}(z_\infty, z'_\infty) \quad \text{and, hence} \quad \lim_{k \to \infty} d_{\tilde{\Sigma}^k_q}(z_k, z'_k) = d_{\tilde{\Sigma}^\infty_q}(z_\infty, z'_\infty) \neq 0.
\]
Equivalently, we get:
\[
\lim_{k \to \infty} d^K_{\tilde{\Sigma}_0}(\tilde{\pi}_k(z_k), \tilde{\pi}_k(z'_k)) = d^K_{\mathbb{B}^{n-1} \times \{0\}}(\tilde{\pi}_\infty(z_\infty), \tilde{\pi}(z'_\infty)).
\]
Since \(d^K_{\tilde{\Sigma}_0}(\tilde{\pi}_k(z_k), \tilde{\pi}_k(z'_k)) = d^K_{\Sigma^q}(z_k, z'_k)\), we conclude that \(d^K_{\mathbb{B}^{n-1} \times \{0\}}(\tilde{\pi}_\infty(z_\infty), \tilde{\pi}(z'_\infty)) \neq 0\) and \(\tilde{\pi}_\infty\) is one-to-one on \(\tilde{\Sigma} \). Let \(\tilde{q}_\infty \in \mathbb{B}^{n-1} \times \{0\}\). For every \(k\), there exists \(q_k \in \tilde{\Sigma}^q\) such that \(\lim_{k \to \infty} \tilde{\pi}_k(q_k) = \tilde{q}_\infty\). Moreover, since \(\tilde{\pi}_k\) is a biholomorphism, we have for every \(k\):
\[
d^K_{\tilde{\Sigma}_0}(\tilde{\pi}_k(q_k), \tilde{\pi}_k(q')) = d^K_{\Sigma^q}(q, q').
\]
Since
\[
\lim_{k \to \infty} \tilde{\pi}_k(q) = \tilde{\pi}_\infty(q) \in \tilde{\Sigma} \infty
\]
and
\[
\lim_{k \to \infty} d^K_{\tilde{\Sigma}_0}(\tilde{\pi}_k(q_k), \tilde{\pi}_k(q)) = d^K_{\mathbb{B}^{n-1} \times \{0\}}(\tilde{\pi}_\infty(q), \tilde{q}_\infty),
\]
there exists \(C > 0\) such that \(\sup_{k \geq 0} d^K_{\tilde{\Sigma}_0}(\tilde{\pi}_k(q_k), \tilde{\pi}_k(q)) \leq C\). This implies:
\[
\sup_{k \geq 0} d^K_{\tilde{\Sigma}_0}(q_k, q) \leq C.
\]
Hence the sequence \((q_k)_k\) is relatively compact in \(\mathbb{B}^n\) and we may extract a subsequence converging to \(q_\infty \in \tilde{\Sigma}^q_\infty\). Since \(\tilde{\pi}_k\) converges to \(\tilde{\pi}_\infty\) uniformly on compact subsets of \(\mathbb{B}^n\), we get \(\tilde{\pi}_\infty(q_\infty) = \tilde{q}_\infty\) and hence \(\tilde{\pi}_\infty\) is onto. \(\square\)

Proposition 3.1 follows now from Proposition 3.5 and Proposition 3.6, setting \(F := \{\tilde{\Sigma}^q_\infty, q \in \mathbb{B}^n\}\) and \(\pi := \tilde{\pi}_\infty\).

References
