THREE MANIFOLD GROUPS, KÄHLER GROUPS
AND COMPLEX SURFACES

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Abstract. Let
\[ 1 \to N \to G \to Q \to 1 \]
be an exact sequence of finitely presented groups, where \( Q \) is infinite and not virtually cyclic, and is the fundamental group of some closed 3-manifold.

If \( G \) is Kähler, we show that \( Q \) contains as a finite index subgroup either a finite index subgroup of the 3-dimensional Heisenberg group or the fundamental group of the Cartesian product of a closed oriented surface of positive genus and the circle. As a corollary, we obtain a new proof of a theorem of Dimca and Suciu in [DS09] by taking \( N \) to be the trivial group.

If \( G \) is the fundamental group of a compact complex surface, we show that \( Q \) must contain the fundamental group of a Seifert-fibered three manifold as a finite index subgroup, and \( G \) contains as a finite index subgroup the fundamental group of an elliptic fibration.

We also give an example showing that the relation of quasi-isometry does not preserve Kähler groups. This gives a negative answer to a question of Gromov which asks whether Kähler groups can be characterized by their asymptotic geometry.

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1. Introduction

In this paper, we shall be concerned with the following general set-up:

\[ 1 \to N \to G \to Q \to 1 \quad (*\) \]

is an exact sequence of finitely presented groups, where \( Q \) is the fundamental group of a closed 3-manifold. We shall further assume that \( Q \) is infinite and not virtually cyclic. The group \( G \) will either be a Kähler group, i.e., the fundamental group of a compact Kähler manifold, or the fundamental group of a compact complex surface. We fix the letters \( N, G, Q, i, q \) to have this connotation throughout this paper. We investigate the restrictions that these assumptions impose on the nature of \( G \) and \( Q \). It will turn out that the existence of such an exact sequence \((*)\) shall force \( Q \) to be the fundamental group of a Seifert-fibered 3-manifold.

The following is the first main theorem of this paper dealing with the case that \( G \) in \((*)\) is Kähler (see Theorem 3.6):

**Theorem 1.1.** Let

\[ 1 \to N \to G \to Q \to 1 \]

be the exact sequence \((*)\) such that \( G \) is a Kähler group. Then there exists a finite index subgroup \( Q' \) of \( Q \) such that either \( Q' \) is a finite index subgroup of the 3-dimensional Heisenberg group or \( Q' = \pi_1(\Sigma \times S^1) \), where \( \Sigma \) is a closed oriented surface of positive genus.

(The 3-dimensional Heisenberg group consists of the unipotent upper triangular elements of \( \text{GL}(3, \mathbb{Z}) \).)

The next theorem deals with the case that \( G \) is the fundamental group of a compact complex surface (see Theorem 5.3).

**Theorem 1.2.** Let

\[ 1 \to N \to G \to Q \to 1 \]

be the exact sequence \((*)\) such that \( G \) is the fundamental group of a compact complex surface. Then there exists a finite index subgroup \( Q' \) of \( Q \) such that \( Q' \) is the fundamental group of a Seifert-fibered 3-manifold with hyperbolic or flat base orbifold. Also there exists a finite index subgroup \( G' \) of \( G \) such that \( G' \) is the fundamental group of an elliptic complex surface \( X \) which is a circle bundle over a Seifert-fibered 3-manifold.

Stronger results when \( X \) is of Class VII, or admits an elliptic fibration, are given in Theorem 4.1 and Proposition 5.2 respectively.

As a consequence of these results we also get the following theorem (see Theorem 6.1):

**Theorem 1.3.** Let \( M \) be a closed orientable 3-manifold. Then \( M \times S^1 \) admits a complex structure if and only if \( M \) is Seifert fibered.

Let \( Q \) be the fundamental group of a closed 3-manifold. Suppose further that \( Q \) is infinite. Donaldson and Goldman conjectured that \( Q \) cannot be a Kähler group. This conjecture was recently proven by Dimca and Suciu [DS09]; later a different proof was given by Kotschick [Kot10]. We obtain a proof of this theorem of [DS09] by setting \( N \) in Theorem 1.1 to be the trivial group (see Theorem 3.7).
In Theorem 1.2, setting $N$ to be the trivial group we conclude that $Q$ is not the fundamental group of a compact complex surface if $Q$ is infinite and not virtually cyclic (see Theorem 5.4).

Finally, we give an example demonstrating the fact that Kählerness is not preserved by quasi-isometries. This gives a strong negative answer to the following question of Gromov (Problem on page 209 of [Gro93]):

Can one characterize Kähler groups by their asymptotic geometry?

The above question makes sense only within the holomorphic category (as the fundamental group of a 2-torus is an index two subgroup of the fundamental group of a Klein bottle). However, if a group $G$ acts freely by holomorphic automorphisms on a simply connected (non-compact) Kähler manifold $X$ and there exists a finite index subgroup $G_1$ of $G$ such that the quotient space $X/G_1$ is Kähler, then $G$ is a Kähler group (cf. [Ara09]).

In the light of this, one makes the above question precise by asking if quasi-isometric to a Kähler group implies commensurability to a (possibly different) Kähler group. This turns out to be false. We give examples of aspherical complex surfaces with quasi-isometric fundamental groups, some of which are Kähler and others not.

The main tools used in this paper are:

1. A detailed structure theory for 3-manifolds arising from the Geometrization Theorem of Thurston and Perelman.
2. The cut-Kähler Theorem of Delzant and Gromov [DG05].
3. A coarse geometric technique of mutual coboundedness for pairs of path-connected subsets of a geodesic metric space.

In particular, our techniques differ from those of [DS09] and [Kot10]. The main difference is that we approach the problem from a 3-manifolds and Geometric Group Theory perspective rather than a Complex Geometry perspective.

2. Kähler Groups, Complex Surfaces and 3 Manifolds

We describe some preliminary material in this section. All manifolds will be connected.

2.1. Restrictions for Kähler Groups. In this subsection, we collect together known restrictions on Kähler groups that are used here. See [ABCKT96] for further details.

The following theorem imposes strong restrictions on homomorphisms from a Kähler group to a real hyperbolic lattice.

**Theorem 2.1** ([DP10], [CT89], [Sam86], Section 6.4 of [ABCKT96]). Let $\Gamma$ be a Kähler group and $\Gamma_1$ be a lattice in $\mathbb{H}^3$. Let $\phi: \Gamma \to \Gamma_1$ be any homomorphism with infinite image. Then either $\phi$ factors through a fibration onto a hyperbolic 2-orbifold or $\phi(\Gamma)$ is cyclic.

The next Theorem imposes strong restrictions on homomorphisms from a Kähler group to solvable groups.
Theorem 2.2. [Bru03] Let \( G \) be a Kähler group such that there is an exact sequence
\[
1 \rightarrow F \rightarrow G \rightarrow H \rightarrow 1,
\]
where \( F \) does not admit a surjective homomorphism onto \( F^\infty \) (the free group on infinitely many generators), and \( H \) is a solvable group of finite rank. Then there exist normal subgroups \( H_2 \subset H_1 \) of \( H \) such that
(a) \( H_1 \) has finite index in \( H \),
(b) \( H_1/H_2 \) is nilpotent, and
(c) \( H_2 \) is torsion.

The following version of the Cut-Kähler Theorem of Delzant and Gromov is one of the principal tools of this paper. For further details see Sections 6, 7 and 3.8 in [DG05].

Theorem 2.3. [DG05] Let \( G \) be the fundamental group \( G \) of a compact Kähler manifold \( X \), and let \( H \subset G \) be a subgroup. Suppose that

(1) the cover \( X_1 \) of \( X \) corresponding to the subgroup \( H \) is non-amenable, i.e., domains in \( X_1 \) satisfy a linear isoperimetric inequality, and
(2) \( \#(\text{Ends}(X_1)) > 2 \).

Then there is a finite cover \( X' \) of \( X \) admitting a surjective holomorphic map \( X' \rightarrow S \) to a Riemann surface \( S \) with connected fibers, such that the pullback to \( \pi_1(X') \subset \pi_1(X) \) of some subgroup in \( \pi_1(S) \) coincides with \( H \cap \pi_1(X') \). In particular, the kernel of the induced homomorphism \( \pi_1(X') \rightarrow \pi_1(S) \) is contained in \( H \cap \pi_1(X') \).

Remark 2.4. The map from \( X \) to \( S \) induces a surjection from \( \pi_1(X) \) to \( \pi_1(S) \). This follows from the well-known fact that a surjective holomorphic map between compact complex manifolds with connected fibers induces a surjection of fundamental groups.

2.2. Restrictions for Complex Non-Kähler Surfaces. We start with the following theorem due to Kodaira.

Theorem 2.5 (See Theorem 1.28, Corollary 1.29 of [ABCKT96]). Let \( \Gamma \) be a finitely presented group. Then \( \Gamma \) is the fundamental group of a Kähler surface if and only if it is the fundamental group of a compact complex surface with even first Betti number.

As in the previous subsection, here we collect together known restrictions on fundamental groups of compact non-Kähler complex surfaces that we will need.

The following Theorem is due to Kodaira; see Theorems 1.27, 1.38 of [ABCKT96], [BHPV04, pp. 244–245], [FM94, Ch. 2]):

Theorem 2.6. Let \( X \) be a compact complex surface with odd first Betti number. Then either \( X \) is elliptic or of Class VII, i.e., \( b_1(X) = 1 \) and the Kodaira dimension of \( X \) is negative.

If \( X \) is minimal of Class VII, and admits non-constant meromorphic functions, then \( X \) is diffeomorphic to \( S^3 \times S^1 \).

If \( X \) is a minimal elliptic surface with odd first Betti number, then the Euler characteristic \( \chi(X) = 0 \) and the universal cover \( \tilde{X} \) is diffeomorphic to \( S^3 \times \mathbb{R} \) or \( \mathbb{R}^4 \).
Corollary 2.7 (Corollary 1.41 of [ABCKT96]). Let $G$ be a group with $b_1(G) = 2m + 1$ with $m \geq 1$ such that $G$ is the fundamental group of a complex surface $X$. Then $X$ is elliptic. If the elliptic fibration $X$ is nonsingular, then $X$ is a $K(G, 1)$ space.

The next three statements impose restrictions on fundamental groups of class VII complex surfaces.

Lemma 2.8 (Lemma 1.45 of [ABCKT96]). If $X$ is a minimal complex surface of Class VII, then any finite-sheeted cover of $X$ is of class VII. Further the intersection form on $H_2(X, \mathbb{Z})/\text{Tor}$ is negative definite.

Corollary 2.9. Let $G$ be a finitely presented group. If $G$ has a finite index subgroup $G'$ such that $b_1(G') > 1$, then $G$ cannot be the fundamental group of a complex surface of Class VII.

Theorem 2.10 ([CT97]). Let $M$ be a non-elliptic compact complex surface with first Betti number one and admitting no nonconstant meromorphic functions. Let $N$ be a compact Riemannian manifold of constant negative curvature. Let $\phi : \pi_1(M) \to \pi_1(N)$ be a homomorphism. Then the image of $\phi$ is either trivial or an infinite cyclic group.

2.3. 3 Manifolds. We very briefly recall the necessary 3-manifold topology and geometry we need and refer to [Hem76], [Bon02], [Sco83a] for details. A closed orientable 3-manifold is said to be prime if it cannot expressed as a non-trivial connected sum (denoted by $\#$) of 3-manifolds. Let $M$ be a closed orientable 3-manifold. Then, there is a unique collection (up to permutation) of prime 3-manifolds $M_1 \ldots M_k$ such that $M$ is homeomorphic to $M_1 \# \ldots \# M_k$. Further $\pi_1(M) = \pi_1(M_1) \ast \ldots \ast \pi_1(M_k)$. Cutting $M$ along essential spheres and capping off the resulting spheres with 3-balls, we get a unique prime decomposition of $M$.

Now suppose that $M$ is prime. Then, up to isotopy, there is a unique compact two dimensional submanifold $T$, each of whose components is a 2-sided essential torus, such that every component of $M \setminus T$ either admits a Seifert fibration or else any essential embedded torus in it is parallel to the boundary. Decomposing a 3-manifold along such tori will be called the torus decomposition.

A consequence of the Geometrization Theorem (due to Thurston and Perelman), [Per02], [Per03a], [Per03b], is that after prime decomposition and torus decomposition, each resulting piece admits a complete geometric structure modeled on $E^3$, $S^3$, $H^3$, $H^2 \times E^1$, $S^2 \times E^1$, $E^2 \times E^1$, $H^2 \times E^1$ or $Sol$. Of these, only manifolds modeled on $S^2 \times E^1$ contains an essential 2-sphere (see [Sco83a]).

Theorem 2.11. ([Bon02], Lemma 2.1 of [KL98]) Let $M$ be a closed, orientable prime 3-manifold with nontrivial torus decomposition. Then there is a finite cover $M_1$ of $M$ such that the any piece in the torus decomposition of $M_1$ is either a complete finite volume hyperbolic manifold or of the form $\Sigma \times S^1$, where $S$ is a compact surface with at least two boundary components and of positive genus.

The following theorem is a consequence of works of Scott, Casson-Jungreis and Gabai as well as Perelman’s proof of the Poincaré Conjecture.

Theorem 2.12 ([CJ94], [Gab92], [Sco83b]). Let $M$ be a compact, orientable, irreducible 3-manifold such that $\pi_1(M)$ contains an infinite cyclic normal subgroup. Then $M$ is a Seifert-fibered space.
The next theorem is a consequence of work of Luecke (Theorem 1 of \cite{Lue88}) and the residual finiteness of (geometrizable) 3-manifold groups \cite{Hem87}.

**Theorem 2.13.** Let $M$ be a compact, orientable, irreducible 3-manifold such that at least one of the following conditions is satisfied:

(a) $M$ contains an incompressible torus and is not a Sol manifold.
(b) $M$ is a Seifert-fibered space such that the base of the Seifert fibration is not an elliptic orbifold.
(c) $M$ is not prime.

Then $M$ has a finite cover $\rho_1 : M_1 \rightarrow M$, such that $\text{rank}(H_1(M_1)) \geq 3$. Further, if $M$ is not covered by the 3-torus, then for any positive integer $k$, there is a finite cover, $\rho_k : M_k \rightarrow M$, such that $\text{rank}(H_1(M_k)) > k$.

Case (a), where $M$ has a non-trivial torus decomposition, is due to Luecke. Case (b) follows from the fact that fundamental groups of Seifert manifolds admit surjections to the fundamental groups of the associated base orbifold \cite{Sco83a}. Case (c) is a consequence of Grushko’s theorem when $M$ is not prime (cf. Lemma 7 of \cite{Kot10}).

Combining Theorem 2.13 with Corollary 2.9, we have the following:

**Corollary 2.14.** Let

$$1 \rightarrow N \rightarrow G \xrightarrow{q} Q \rightarrow 1$$

be an exact sequence of groups, where $Q$ is the fundamental group of a closed 3-manifold $M$, and $N$ is finitely presented. Suppose further that at least one of the following conditions is satisfied:

(a) $M$ contains an incompressible torus and is not a Sol manifold.
(b) $M$ is a Seifert-fibered space such that the base of the Seifert fibration is not an elliptic orbifold.
(c) $M$ is not prime.

Then $G$ cannot be the fundamental group of a complex surface of class VII.

**Proof.** This is a consequence of the following simple fact: Let $A$ and $B$ be topological spaces and $q : \pi_1(A) \rightarrow \pi_1(B)$ a surjective homomorphism. If $B_1$ is a finite cover of $B$ with positive $b_1$, then the finite cover $A_1$ of $A$ corresponding to the subgroup $q^{-1}(B_1)$ also has positive $b_1$. \qed

2.4. **Non-amenability and Coboundedness.** In this subsection, we shall focus on 3-manifolds admitting a non-trivial torus decomposition.

**Structure of Ends:** Let $L_1$ be a non-compact hyperbolic 3-manifold of finite volume, and let $L$ be the manifold with boundary obtained from $L_1$ by removing the interiors of cusps. Let $H_0$ be the torus subgroup of $\pi_1(L)$ corresponding to a cusp. Let $L_T$ denote the cover of $L$ corresponding to the torus subgroup $H_0$. Then $L_T$ is non-amenable; an easy way to see this is to use the fact that $\pi_1(L)$ is strongly hyperbolic relative to the collection of cusp subgroups \cite{Far98}.

Similarly, if $\Sigma$ is a compact 2-manifold with boundary and genus greater than one, let $H_0$ denote the cyclic subgroup of $\pi_1(\Sigma)$ corresponding to a boundary component. Then
π₁(Σ) is strongly hyperbolic relative to $H₀$ (cf. Proposition 2.10 of [Mj08]). Hence the cover of Σ corresponding to $H₀$ is non-amenable. This implies further that the cover of $Σ × S¹$ corresponding to $H₀ ⊕ ℤ$ is non-amenable.

Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. By passing to a finite cover, if necessary, and using Theorem 2.11, we can assume that each piece of the torus decomposition of $M$ is either a non-compact complete hyperbolic manifold of finite volume or of the form $Σ × S¹$, with Σ a compact surface with non-empty boundary and genus at least two.

Let $M₁$ be a piece of the torus decomposition. Note that each boundary component of $M₁$ is a torus. Then the cover of $M$ corresponding to the subgroup $π₁(M₁) ⊂ π₁(M)$ has ends corresponding to the different torus boundary components of $M₁$. Since each end of this cover is non-amenable by the above discussion we have the following lemma.

**Lemma 2.15.** Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. Let $M₁$ be a piece of the torus decomposition. Then the cover of $M$ corresponding to the subgroup $π₁(M₁) ⊂ π₁(M)$ is non-amenable.

**Coboundedness:**

**Definition 2.16.** Let $(Y,d)$ be a metric space. We say that two subsets $Y₁,Y₂ ⊂ Y$ are mutually cobounded if for all $D ≥ 0$, the subset

$$\{(y₁,y₂) ∈ Y₁ × Y₂ \mid d(y₁,y₂) ≤ D\}$$

is compact.

See Section 1 of [MS09] for a closely related definition in the context of hyperbolic metric spaces.

The next two lemmas give standard examples of cobounded subsets of hyperbolic metric spaces (see [Far98], [Bow97] or [Mj06, Section 2.2] for proofs).

**Lemma 2.17.** Let $M$ be a non-compact complete hyperbolic $n$-manifold of finite volume. Let $H₁$ and $H₂$ be lifts of a neighborhood of a cusp in $M$ to the universal cover $\widetilde{M}$. Then $H₁$ and $H₂$ are cobounded horoballs in $\widetilde{M}$. Equivalently, $∂H₁$ and $∂H₂$ are cobounded horospheres in $\widetilde{M}$.

**Lemma 2.18.** Let $Σ$ be a hyperbolic surface with (possibly empty) totally geodesic boundary. Let $σ₁$ and $σ₂$ be lifts of a closed geodesic in $Σ$ to the universal cover $\widetilde{Σ}$. Then $σ₁$ and $σ₂$ are cobounded in $\widetilde{Σ}$.

Now let $N = Σ × S¹$, where Σ is an orientable surface with boundary, having genus greater than one. Equip $N$ with a product metric. It follows from Lemma 2.18 that any two lifts $σ₁$ and $σ₂$ of the boundary components of Σ to the universal cover $\widetilde{N}$ are cobounded. Let $Eᵢ = σᵢ × ℜ$ denote the corresponding lift of bounding tori of $N$ to $\widetilde{N}$ containing $σᵢ$. Thus we have the following:

**Corollary 2.19.** Let $N = Σ × S¹$, where Σ is an orientable surface with boundary, having genus greater than one. Equip $N$ with a product Riemannian metric, and denote by $d_{\widetilde{Σ}}$ the induced distance function on $\widetilde{Σ}$. Let $σ₁$ and $σ₂$ be lifts of the boundary components of
\[ \Sigma \text{ to the universal cover } \tilde{\Sigma}. \text{ Let } E_i = \sigma_i \times \mathbb{R}, i = 1, 2, \text{ denote the corresponding lifts of bounding tori of } N \text{ to } \tilde{\Sigma} \text{ containing } \sigma_i. \text{ Let } p_i \in \sigma_i \text{ be such that} \]

\[ d_{\Sigma}(p_1, p_2) = d_{\Sigma}(\sigma_1, \sigma_2). \]

Then for all \( D \geq 0 \), there exists \( \epsilon > 0 \) such that

\[ \{(y_1, y_2) \in E_1 \times E_2 | d(y_1, y_2) \leq D\} \subset (B_{\epsilon}(p_1) \times \mathbb{R}) \times (B_{\epsilon}(p_2) \times \mathbb{R}), \]

where \( B_{\epsilon}(p_i) \) denotes the \( \epsilon \)-neighborhood of \( p_i \) in \( \sigma_i \).

Let, as at the beginning of this subsection, \( M \) be a prime 3-manifold admitting a non-trivial torus decomposition. Let \( M_1 \) be a piece of the torus decomposition. Let \( \tilde{M} \) denote the universal cover of \( M \) equipped with some equivariant Riemannian metric. The bounding tori of \( M_1 \) lift to copies of the Euclidean plane in \( \tilde{M} \). Let \( \{E_\alpha\} \) be the collection of lifts of tori \( (\tilde{M}) \) of all tori in the torus decomposition of \( M \). If \( M_1 \) is a hyperbolic piece, then by Lemma 2.17, it follows that any pair \( E_1, E_2 \in \{E_\alpha\} \) which are lifts of bounding tori of \( M_1 \) is cobounded in \( \tilde{M} \). In fact since the lifts of any hyperbolic piece of the torus decomposition separate \( \tilde{M} \), it follows that there exist \( E_1, E_2 \in \{E_\alpha\} \) separated by such a lift. Hence we have:

**Lemma 2.20.** Let \( M \) be a prime 3-manifold admitting a non-trivial torus decomposition such that at least one of the pieces of the torus decomposition is hyperbolic. Let \( \{E_\alpha\} \) be the collection of lifts (to \( \tilde{M} \)) of all tori in the torus decomposition of \( M \). There exist \( E_1, E_2 \in \{E_\alpha\} \) which are cobounded in \( \tilde{M} \).

Next assume that \( M \) is a prime 3-manifold admitting a non-trivial torus decomposition such that all pieces of \( M \) are Seifert-fibered, i.e., \( M \) is a graph manifold. By passing to a finite-sheeted cover if necessary we may assume, by Theorem 2.11, that each piece of the torus decomposition is of the form \( \Sigma \times S^1 \), where \( \Sigma \) is a compact surface with boundary and genus greater than one.

In the universal cover \( \tilde{M} \), let \( \tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4 \) be a sequence of lifts of Seifert-fibered pieces such that \( \tilde{M}_i \cap \tilde{M}_{i+1} = E_i \), for \( i = 1, 2, 3 \), where \( E_i \) is the universal cover of a torus \( T_i \) in \( M \). We shall show that \( E_1 \) and \( E_3 \) satisfy the conclusions of Lemma 2.20.

Let \( M_i = \Sigma_i \times \alpha_i \) where \( \alpha_i \) is a unit circle. Assume without loss of generality (by passing to finite-sheeted covers if necessary) that \( \{M_i\}_{i=1}^4 \) are embedded submanifolds of \( M \) and that \( M_i \cap M_{i+1} = T_i \), for \( i = 1, 2, 3 \), where \( T_i \) is an embedded essential torus in \( M \) appearing in the torus decomposition. Also, \( T_i = \sigma_i \times \alpha_i \) where \( \sigma_i \) is a boundary curve of \( \Sigma_i \) and \( \alpha_i \) is the circle fiber.

By Corollary 2.19 the set of points \( x \in E_1 \) and \( y \in E_2 \) with \( d(x, y) \leq D \) must lie within \( (B_{\epsilon}(p_1) \times \alpha_2) \times (B_{\epsilon}(p_2) \times \alpha_2) \), where \( [p_1, p_2] \) denotes the geodesic in \( \Sigma_2 \) joining the appropriate lifts of \( \sigma_2 \). The same argument shows that the set of points \( u \in E_2 \) and \( v \in E_3 \) with \( d(u, v) \leq D \) must lie within \( (B_{\epsilon}(q_1) \times \alpha_3) \times (B_{\epsilon}(q_2) \times \alpha_3) \), where \( [q_1, q_2] \) denotes the geodesic in \( \Sigma_3 \) joining the appropriate lifts of \( \sigma_3 \). Next, \( (B_{\epsilon}(p_2) \times \alpha_2) \) and \( (B_{\epsilon}(q_1) \times \alpha_3) \) are cobounded as the fibers of \( M_2 \) and \( M_3 \) do not agree, being different Seifert components. Hence we conclude that \( E_1 \) and \( E_3 \) are cobounded in \( \tilde{M} \), in other words, \( E_1 \) and \( E_3 \) satisfy the conclusions of Lemma 2.20 as claimed above.

Combining this with Lemma 2.20, we conclude:
Proposition 2.21. Let $M$ be a prime 3-manifold admitting a non-trivial torus decomposition. Let $\{E_\alpha\}$ be the collection of lifts (to $\widetilde{M}$) of all tori in the torus decomposition of $M$. There exist $E_1, E_2 \in \{E_\alpha\}$ which are cobounded in $\widetilde{M}$.

Infinite normal subgroups offer the opposite situation, as shown by the following lemma.

Lemma 2.22. Let $M$ be a topological space and $A \subset M$ an incompressible subspace (i.e., $i : A \longrightarrow M$ induces an injective map $i_* : \pi_1(A) \longrightarrow \pi_1(M)$). Suppose $N$ is an infinite normal subgroup of $\pi_1(M)$ contained in $\pi_1(M, A)$, where $\pi_1(M)$ is the group of deck transformations of $\widetilde{M}$ and $\widetilde{M}$ has been endowed with a $\pi_1(M)$-equivariant distance function $d$.

Proof. It is enough to prove that $\widetilde{A}$ and $g\widetilde{A}$ are not cobounded for any $g \in \pi_1(M)$. Let $\{n\}_{n=1}^\infty \subset N$ be an infinite subset and let $s_n := g_n g^{-1} \in N$. Fix $x \in \widetilde{A}$. Then $s_n x \in \widetilde{A}$, $g_n x \in \widetilde{A}$ and $d(s_n x, g_n x) = d(x, gx)$. Since $\{g_n x\}$ is a noncompact subset of $\widetilde{M}$, we see that $\widetilde{A}$ and $g\widetilde{A}$ are not cobounded by taking $D = d(x, gx)$. \qed

3. Kähler Groups

In this Section we use the restrictions described in Section 1 to rule out various possibilities. In this section, the following possibilities will be taken up and ruled out one-by-one.

(a) $M$ admits a non-trivial prime decomposition.
(b) $M$ admits a non-trivial torus decomposition.
(c) $M$ admits a Sol geometric structure.
(d) $M$ admits a hyperbolic structure.

Spherical or elliptic 3-manifolds have finite fundamental group and 3-manifolds with $S^2 \times \mathbb{R}$ geometry have virtually cyclic fundamental group. These are ruled out by the hypothesis on $Q$. Thus at the end of the discussion we shall conclude that $M$ is a Seifert-fibered space with Euclidean or hyperbolic base orbifold. We shall then proceed to extract further restrictions on what $Q$ may be in case $M$ is Seifert-fibered.

We will use the following simple Lemmas for (a) and (b):

Lemma 3.1. If

$$1 \longrightarrow N_0 \longrightarrow \pi_1(S) \longrightarrow Q \longrightarrow 1$$

is an exact sequence of finitely generated groups such that $S$ is a closed orientable surface and $Q$ is an infinite non-virtually cyclic fundamental group of a 3-manifold, then $N_0$ cannot be finitely generated.

Proof. If $S$ has genus $g > 1$ then the only finitely generated normal subgroup of such a $\pi_1(S)$ is the trivial group. This forces $Q$ and $\pi_1(S)$ to be commensurable, an impossibility.

For a torus, $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}$ and again, such an exact sequence is impossible as it forces $Q$ and $\pi_1(S)$ to be commensurable. \qed
Lemma 3.2. Let $X$, $Y$ be compact Riemannian manifolds and $q : \pi_1(X) \to \pi_1(Y)$ a surjective homomorphism. If $Y_1$ is a cover of $Y$ and $X_1$ the corresponding cover of $X$ (with $\pi_1(X_1) = q^{-1}(\pi_1(Y_1))$), then $X_1$ is quasi-isometric to $Y_1$.

Proof. This follows from the fact that the cover of $X$ corresponding to the subgroup $\ker(q) \subset \pi_1(X)$ is quasi-isometric to the universal cover of $Y$. □

$M$ admits a non-trivial prime decomposition

Let $G$ be as in (*) such that $Q$ is the fundamental group of a 3-manifold $M$ that admits a non-trivial prime decomposition.

Let, if possible, $X$ be a Kähler manifold with fundamental group $G$. Consider the cover $X_1$ of $X$ corresponding to the normal subgroup $N$. Note that universal cover $\tilde{M}$ of $M$ has infinitely many ends and is non-amenable. Since $X_1$ is quasi-isometric to $\tilde{M}$ by Lemma 3.2, $X_1$ has the same properties and we can apply Theorem 2.3 by taking $H = N$. Therefore there is a holomorphic map $\psi$ from a finite cover of $X$ to a Riemann surface $S$ with connected fibers such that the pullback to $\pi_1(X)$ of some subgroup in $\pi_1(S)$ equals $N$. In particular the kernel of $\psi$ is a normal subgroup of $G$ contained in $N$. Hence we have an exact sequence (the surjectivity of the last map follows from Remark 2.4)

$$1 \to \frac{N}{\ker(\psi)} \to \pi_1(S) \to Q \to 1$$

where $\frac{N}{\ker(\psi)}$ is a finitely generated normal subgroup of $\pi_1(S)$ and $Q$ is a non-trivial non-virtually cyclic free product. This is impossible by Lemma 3.1.

Remark 3.3. Note that we have not used the fact that $Q$ is a 3-manifold group here, but only that it is a non-trivial non-virtually cyclic free product admitting finite index subgroups.

$M$ admits a non-trivial torus decomposition

Let $G$ be as in (*) such that $Q$ is the fundamental group of a 3-manifold $M$ that admits a non-trivial torus decomposition.

By Theorem 2.11 we can pass to a finite-sheeted cover of $M$ such that all the Seifert-fibered components of the torus decomposition are of the form $\Sigma \times S^1$, where $\Sigma$ is an orientable compact surface of genus $g > 1$, with boundary. By passing to a further finite-sheeted cover we may also assume that some connected component $M_1 = \Sigma_1 \times S^1$ of the torus decomposition $\Sigma_1$ has more than two boundary components.

Let, if possible, $X$ be a Kähler manifold with fundamental group $G$. Let

$$H := q^{-1}(\pi_1(M_1)) \subset G,$$

and let $X_1$ be the covering of $X$ corresponding to the subgroup $H$. If $M'$ denotes the cover of $M$ corresponding to the subgroup $\pi_1(M_1) \subset \pi_1(M)$, then $M'$ has at least three ends because $M_1$ has more than two boundary tori. Moreover $M'$ is non-amenable by Lemma 2.15. Since $M'$ and $X_1$ are quasi-isometric by Lemma 3.2, the Kähler manifold $X_1$ has these properties as well. We can then apply the cut-Kähler Theorem 2.3 of
Delzant-Gromov to conclude that there exists a finite cover $\psi$ from (a finite cover of) $X$ to a Riemann surface $S$ with connected fibers such that the pullback to $\pi_1(X)$ of some subgroup in $\pi_1(S)$ equals $H$. In particular, the kernel $\ker(\psi_*) \subset H$.

Let $\tilde{M}$ denote the universal cover of $M$ equipped with some equivariant Riemannian metric. It follows from Proposition 2.21 that there exist two lifts $\tilde{M}_1, \tilde{M}_2$ of $M_1$ which are separated by lifts $E_1, E_2$ of tori (occurring in the torus decomposition of $M$) that are cobounded. Hence the lifts $\tilde{M}_1$ and $\tilde{M}_2$ are themselves cobounded, meaning for any $D > 0$, the collection of points $x \in \tilde{M}_1$ and $y \in \tilde{M}_2$ with $d(x, y) \leq D$ is compact. Hence by Lemma 2.22 the normal subgroup $q(\ker(\psi_*)) \subset \pi_1(M_1) \subset Q$ must be finite. But any finite normal subgroup of $Q$ has to be trivial. Hence $\ker(\psi_*) \subset N$. We therefore have an exact sequence

$$1 \longrightarrow N \longrightarrow \ker(\psi_*) \longrightarrow \pi_1(S) \longrightarrow Q \longrightarrow 1$$

where $N = \ker(\psi_*)$ is a finitely generated normal subgroup of $\pi_1(S)$ and $Q = \pi_1(M)$. This is impossible by Lemma 3.1.

**$M$ is Sol**

Let $G$ be as in (*) such that $Q$ is the fundamental group of a 3-manifold $M$ which is Sol.

Since the group $G$ admits a surjection to the solvable non-nilpotent group $\pi_1(M)$, Theorem 2.2 shows that $G$ cannot be Kähler.

**$M$ hyperbolic**

Let $G$ be as in (*) such that $Q$ is the fundamental group of a 3-manifold $M$ which is hyperbolic.

Since the group $G$ admits a surjection to a closed hyperbolic 3-manifold group $Q$, $G$ cannot be Kähler by Theorem 2.1.

By the above discussion, it follows that if

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is an exact sequence of groups such that
(a) $Q = \pi_1(M)$ is the fundamental group of a closed 3-manifold $M$,
(b) $Q$ is infinite and not virtually cyclic,
(c) $G$ is Kähler, and
(d) $N$ is finitely presented,
then $M$ is Seifert-fibered with base orbifold Euclidean or hyperbolic.

**$M$ Seifert-fibered**

Let $G$ be as in (*) such that $Q$ is the fundamental group of a 3-manifold $M$ which is Seifert-fibered.
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Given a surjective homomorphism \( h : G \to G_1 \) with kernel \( K \), there is an Euler class obstruction \( e(h) \) to the existence of a section of \( h \) associated to the exact sequence

\[
1 \to K \to G \to G_1 \to 1.
\]

A recent Theorem of Arapura (Corollary 5.5 of [Ara09]) asserts the following.

**Theorem 3.4.** If a Kähler group \( G \) admits a surjective homomorphism \( h \) to a surface group \( G_1 = \pi_1(S) \) of genus \( g \) greater than one with \( g \) maximal, then the Euler class \( e(h) \in H^2(G_1, [K,K]) \) is torsion, where \( K \) is the kernel of \( h \).

For a 3-manifold \( M \) which is a twisted (non-zero Euler class) circle bundle over a closed hyperbolic orbifold, some cover is a twisted circle bundle over a closed surface \( S \) of genus greater than one. By abusing notation slightly we call this cover \( M \) and its fundamental group \( Q \). Let \( G_1 = \pi_1(S) \). If

\[
1 \to N_0 (= \mathbb{Z}) \to Q \xrightarrow{\phi} G_1 \to 1
\]

is the associated exact sequence, then \( e(\phi) \) is non-torsion. We have the following.

**Proposition 3.5.** Let

\[
1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1
\]

be an exact sequence of groups such that

(a) \( Q = \pi_1(M) \) is the fundamental group of a Seifert-fibered closed 3-manifold \( M \) with hyperbolic base-orbifold,

(b) \( G \) is a Kähler group, and

(c) \( N \) is finitely presented.

Then a finite-sheeted cover of \( M \) is a product \( \Sigma \times S^1 \), where \( \Sigma \) is a closed oriented surface of genus greater than one.

**Proof.** If \( G \) is Kähler, any finite index subgroup of \( G \) is Kähler. Let \( G = \pi_1(X) \), where \( X \) is a Kähler manifold. By passing to a finite-sheeted cover of \( X \) if necessary, we may assume that \( M \) is a circle bundle over a closed surface \( S \) of genus greater than one. Abusing notation slightly again, call this cover \( X \). We have exact sequences

\[
1 \to N \xrightarrow{i} \pi_1(X) \xrightarrow{q} Q (= \pi_1(M)) \to 1
\]

and

\[
1 \to N_0 (= \mathbb{Z}) \to Q \xrightarrow{\phi} G_1 (= \pi_1(S)) \to 1.
\]

This gives rise to a surjection \( \phi \circ q : \pi_1(X) \to G_1 (= \pi_1(S)) \). If \( e(\phi) \) is non-zero, it follows that \( e(\phi \circ q) \) is non-torsion. Therefore \( e(\phi) = 0 \) by Theorem 3.4. \( \square \)

Hence we have the first main theorem of our paper.

**Theorem 3.6.** Let

\[
1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1
\]

be an exact sequence of groups such that

(a) \( Q = \pi_1(M) \) is the fundamental group of a closed 3-manifold \( M \) such that \( Q \) is infinite and not virtually cyclic,
(b) \(G\) is a Kähler group, and
(c) \(N\) is finitely presented.

Then there exists a finite index subgroup \(Q'\) of \(Q\) such that either \(Q'\) is a finite index subgroup of the 3-dimensional Heisenberg group or \(Q' = \pi_1(\Sigma) \times S^1\), where \(\Sigma\) is a closed oriented surface of positive genus.

**Proof.** Proposition 3.5 deals with the case that the base orbifold is hyperbolic. If the base orbifold is Euclidean, then any twisted bundle over the torus has fundamental group isomorphic to (a finite index subgroup of) the 3-dimensional Heisenberg group. 

If \(N\) is trivial and \(G\) is a Kähler group, then the exact sequence (\(\ast\)) implies that \(Q\) is a Kähler group. But \(b_1(\pi_1(\Sigma \times S^1))\) is odd for any closed oriented surface \(\Sigma\), hence \(\pi_1(\Sigma \times S^1)\) is not Kähler. Also, no finite index subgroup of the 3-dimensional Heisenberg group can be Kähler (Example 3.31 in [ABCKT96, p. 40]). Hence, in view of Theorem 3.6, we have a new proof of the following theorem of Dimca-Suciu (conjectured originally by Donaldson and Goldman).

**Theorem 3.7** ([DS09], [Kot10]). Let \(Q\) be the fundamental group of a closed 3-manifold. Suppose further that \(Q\) is infinite and not virtually cyclic. Then \(Q\) cannot be a Kähler group.

4. **Class VII Surfaces**

Let

\[
1 \longrightarrow N \overset{i}{\longrightarrow} G \overset{q}{\longrightarrow} Q \longrightarrow 1
\]

be an exact sequence of groups where \(Q\) is an infinite, non-virtually cyclic fundamental group of a closed 3-manifold \(M\). Suppose further that \(G\) is the fundamental group of a Class VII surface. No assumptions are made on \(N\) for the purposes of this section. Then by Corollary 2.14 \(M\) is prime and does not admit a non-trivial torus decomposition. Also \(M\) cannot be Seifert fibered over a hyperbolic or flat orbifold.

One can assume that \(X\) is minimal (as blowing down does not change fundamental group). Then either \(X\) is a Hopf surface or else it does not admit any non-constant meromorphic functions.

\(X\) cannot be Hopf as Hopf surfaces have infinite cyclic fundamental group and \(\pi_1(M)\) is infinite, and not virtually cyclic. Hence \(X\) does not admit any non-constant meromorphic functions.

The quotient map from \(G\) to \(Q\) is surjective. Hence, by Theorem 2.10, the manifold \(M\) cannot be hyperbolic.

Finally we dispose of the case that \(Q\) is \textit{Sol}. This is ruled out by the following adaptation of an argument of Kotschick [Kot10] that we reproduce here.

Since \(\pi_1(X)\) surjects onto \(\pi_1(M)\) and the latter fibers over the circle, it follows that the classifying map \(\phi_q\) for \(q : \pi_1(X) \longrightarrow \pi_1(M)\) induces an isomorphism

\[
\phi^*_{q,1} : H^1(M) \longrightarrow H^1(X)
\]
and an injective map $\phi^*_q: H^2(M) \to H^2(X)$. Let $\alpha$ be a generator of $H^2(M, \mathbb{Z})$. Then $\alpha \cup \alpha = 0$ and so $\phi^*_q\alpha \cup \phi^*_q\alpha = 0$, which makes the intersection form indefinite. This contradicts Lemma 2.8. We summarize our conclusions as follows.

**Theorem 4.1.** Let

$$1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1$$

be an exact sequence of groups such that $Q = \pi_1(M)$ is the fundamental group of a closed 3-manifold $M$ which is infinite and not virtually cyclic. Then $G$ cannot be the fundamental group of a Class VII complex surface.

5. **Elliptic Fibrations**

Let

$$1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1$$

be an exact sequence of groups where $Q$ is an infinite, non-virtually cyclic fundamental group of a closed 3-manifold $M$. Suppose further that $G$ is the fundamental group of a compact complex surface $X$ admitting an elliptic fibration. Assume that $N$ is finitely generated.

If the fibration is singular, then the inclusion of the fiber subgroup $\mathbb{Z} \oplus \mathbb{Z}$ into $\pi_1(X)$ has non-trivial kernel and $\pi_1(X)$ must have rational cohomological dimension at most 3. In fact, by Theorem 2.3 of [FM94], existence of singular fibers forces $\pi_1(X)$ to be equal to the fundamental group of the base orbifold of complex dimension one. Hence we have an exact sequence

$$1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1$$

where $G$ is the fundamental group of an orbifold of complex dimension one and $N$ is a finitely generated normal subgroup of $G$. If the orbifold is hyperbolic, the finitely generated normal subgroup $N$ must be finite and $Q$ must therefore have rational cohomological dimension two. But the assumptions on $Q$ force it to have cohomological dimension 3.

If the orbifold is Euclidean, then after passing to a finite index subgroup if necessary, we have an exact sequence

$$1 \to N \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{q} Q \to 1.$$

This forces $Q$ to have cohomological dimension at most two. A contradiction again.

Hence the elliptic fibration must be non-singular (with possibly multiple fibers) and we have an exact sequence (after passing to a finite cover again if necessary)

$$1 \to N_0 (= \mathbb{Z} \oplus \mathbb{Z}) \xrightarrow{\phi} G \xrightarrow{\psi} \pi_1(S) \to 1,$$

where $S$ is a closed orientable surface of positive genus. In particular by Corollary 2.7, the group $G$ admits a closed aspherical 4-manifold $X$ as a $K(G, 1)$. Further, there is a holomorphic fiber bundle structure on $X$ realizing the exact sequence (1) with one dimensional complex tori as fibers. The universal cover of $X$ is homeomorphic to $\mathbb{R}^4$.

We first show that $M$ has to be prime. Assume that $M$ is not prime. Then $Q$ is a non-trivial free product that is not virtually cyclic. Hence the abelian normal subgroup
$q(N_0) \subset Q$ must be trivial, i.e., $N_0 = \mathbb{Z} \oplus \mathbb{Z}$ must be contained in $N$. Therefore we have an exact sequence

$$1 \rightarrow \frac{N}{N_0} \rightarrow \pi_1(S) \rightarrow Q \rightarrow 1,$$

where $\frac{N}{N_0}$ is a finitely generated normal subgroup of $\pi_1(S)$, and $Q$ is a non-trivial free product. This is again impossible by Lemma 3.1. Hence $M$ is prime.

Since $\pi_1(M)$ is not virtually cyclic, it follows that $Q$ is a $PD(3)$ group by the prime and torus decomposition theorems. We shall need the following Theorem of Bieri and Eckmann.

**Theorem 5.1 ([BE73]).** Let $1 \rightarrow K \rightarrow H \rightarrow L \rightarrow 1$ be a short exact sequence of groups. If $K$ and $L$ are duality groups of dimensions $n$ and $m$ respectively, then $H$ is a duality group of dimension $(m + n)$.

We shall try to understand the structure of $M$ in terms of the rank of $q(N_0)$. The rank of $q(N_0)$ is zero, one, or two.

**Case 1: Rank of $q(N_0)$ is zero.** Then $N \cap N_0$ has rank 2 and we have an exact sequence

$$1 \rightarrow N \cap N_0 \rightarrow G' \xrightarrow{q} Q \rightarrow 1,$$

where $G'$ is a finitely presented subgroup of $G$. The exact sequence forces $G'$ to have rational cohomological dimension 5 by Theorem 5.1. Since $G$ has cohomological dimension 4, this is impossible.

**Case 2: Rank of $q(N_0)$ is one.** Since the rank of $q(N_0)$ is one, then $Q$ has a normal $\mathbb{Z}$ subgroup. Hence $M$ is a Seifert-fibered space by by Theorem 2.12. Further $N \cap N_0$ has rank one and so there exists an infinite cyclic subgroup $N_1$ of finite index in $N \cap N_0$. Hence we have an exact sequence

$$1 \rightarrow N_1 \rightarrow G' \xrightarrow{q} Q \rightarrow 1,$$

for some subgroup $G'$ of $G$. This forces $G'$ to be a Poincaré duality group of dimension 4 by Theorem 5.1 and so $G'$ is of finite index in $G$. Hence a finite-sheeted cover of $X$ is a circle bundle over a Seifert-fibered space.

**Case 3: Rank of $q(N_0)$ is two.** Then $Q$ has a normal $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. Hence $M$ is virtually a torus bundle over the circle [Hem76]. Further, $N \cap N_0 = \{1\}$. Hence $N \cap N_0$ is a normal subgroup of $G$ and for all $n \in N, m \in N_0, mn = nm$. Also, $\frac{G}{N \cap N_0} = \mathbb{Z}$. Since $G$ is torsion-free, so is $N$. Also, $N$ must be infinite as $G$ has rational cohomological dimension 4 and $Q$ has rational cohomological dimension 3. Let $H$ be any infinite cyclic subgroup of $N$. Then $HN_0$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and we have a short exact sequence

$$1 \rightarrow HN_0 \rightarrow G' \xrightarrow{q} H_1(= \mathbb{Z}) \rightarrow 1 \quad (2)$$

for some subgroup $G'$ of $G$. This forces $G'$ to be a Poincaré duality group of dimension 4 by Theorem 5.1 and so $G'$ is of finite index in $G$. Further it forces $H$ to have finite index in $N$ and hence $N$ is infinite cyclic. Also the exact sequence in (2) must split and hence by passing to a double cover if necessary we may assume that if $s : H_1 \rightarrow G'$ is a section, then for all $h \in H$ and $h_1 \in s(H_1)$, $hh_1 = h_1h$. Therefore, $X$ is virtually both a 3-torus bundle over the circle as well as a 2-torus bundle over a 2-torus. By passing to
a finite-sheeted cover if necessary, we may assume that $N$ is central. Since each element of $N$ commutes with each element of $N_0$ as well as each element of $s(H_1)$, it follows that $G' = H \oplus Q'$, where $Q'$ is the fundamental group of a 2-torus bundle over the circle.

If $M$ has Sol geometry, then $Q'$ is a Sol group also and hence $b_1(G') = 1 + b_1(Q') = 2$, which in turn implies that the cover of $X$ with fundamental group $G''$ is Kähler. This contradicts Theorem 3.6. Let $M'$ be the cover of $M$ corresponding to $Q'$. Hence the monodromy of the torus bundle $M'$ is either reducible or periodic. In either case, $M'$ and hence $M$ is Seifert-fibered.

We summarize the discussion as follows.

**Proposition 5.2.** Let

$$1 \to N \to G \to Q \to 1$$

be an exact sequence of groups where $Q = \pi_1(M)$ is an infinite non-virtually cyclic fundamental group of a closed 3-manifold $M$, $G$ is the fundamental group of an elliptic complex surface and $N$ is finitely generated. Then

1. $N$ must be virtually infinite cyclic, and
2. $M$ must be a Seifert-fibered space whose base orbifold is flat or hyperbolic and a finite sheeted cover of $X$ must be a circle bundle over a finite sheeted cover of $M$.

Combining Proposition 5.2 with Theorem 4.1 we have the second main theorem of our paper.

**Theorem 5.3.** Let

$$1 \to N \to G \to Q \to 1$$

be an exact sequence of groups such that

(a) $Q = \pi_1(M)$ is the fundamental group of a closed 3-manifold $M$ such that $Q$ is infinite and not virtually cyclic,

(b) $G$ is the fundamental group of a compact complex surface, and

(c) $N$ is finitely presented.

Then there exists a finite index subgroup $Q'$ of $Q$ such that $Q'$ is the fundamental group of a Seifert-fibered 3-manifold with hyperbolic or flat base. Also, there exists a finite index subgroup $G'$ of $G$ such that $G'$ is the fundamental group of an elliptic complex surface $X$ which is a circle bundle over a Seifert-fibered 3-manifold.

Combining Theorem 5.3 with Proposition 5.2 and using the fact that Seifert fibered spaces with hyperbolic or flat base have finite sheeted covers with $b_1 > 1$, we obtain the following theorem of Kotschick.

**Theorem 5.4** (Kot10). Let $Q$ be the fundamental group of a closed 3-manifold. Suppose further that $Q$ is infinite and not virtually cyclic. Then $Q$ cannot be the fundamental group of a compact complex surface.

6. **Further Consequences**

6.1. **Products of 3 Manifolds and Circles.** Let

$$1 \to N \to G \to Q \to 1$$
be an exact sequence of groups where \( N \) is finitely presented, \( Q = \pi_1(M) \) is the fundamental group of a closed Seifert-fibered 3-manifold \( M \) whose base orbifold is flat or hyperbolic, and \( G \) is the fundamental group of a compact complex surface. By Theorem 5.3, \( X \) must be an aspherical elliptic complex surface and \( N \) must be virtually infinite cyclic. We now briefly discuss complex structures on circle bundles over \( M \), where \( M \) is Seifert-fibered.

**Theorem 6.1.** Let \( M \) be a closed orientable 3 manifold. Then \( M \times S^1 \) admits a complex structure if and only if \( M \) is Seifert fibered.

**Proof.** We will first show that every circle bundle over a Seifert-fibered 3-manifold admits a complex structure (cf. [Woo90]). We include a proof of it for completeness.

Let \( X \) be a compact connected Riemann surface, and let \( T = \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z}) \) be the elliptic curve. Consider the short exact sequence of sheaves on \( X \)

\[
0 \to \mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z} \to \mathcal{O}_X \to T \to 0,
\]

where \( T \) is the sheaf of holomorphic functions to \( T \), and \( \mathcal{O}_X \) is the sheaf of holomorphic functions. Since \( H^2(X, \mathcal{O}_X) = 0 \) (as \( \dim \mathbb{C} X = 1 \)), the homomorphism

\[
H^1(X, T) \to H^2(X, \mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z})
\]

in the long exact sequence of cohomologies associated to the short exact sequence in (3) is surjective. This implies that any \( S^1 \times S^1 \)-bundle over \( X \) admits a complex structure.

Let \( M \to X \) be a Seifert-fibered 3-manifold with (base) orbifold points \( \{ p_i \}_{i=1}^n \); let \( m_i \) be the order of \( p_i \). Then a circle bundle over \( M \) is diffeomorphic to a manifold obtained by performing logarithmic transformations on a holomorphic principal \( T \)-bundle over \( X \); the logarithmic transformations are done over the points \( p_i \) using the automorphism of \( T \) given by the automorphism of \( \mathbb{C} \) defined by \( z \mapsto z + 1/m_i \). (See [BHPV04, Ch. V, § 13] for logarithmic transformation.) Therefore, we conclude that every circle bundle over a Seifert-fibered surface admits a complex structure.

The converse direction follows from Theorems 3.6, 5.3 and Proposition 5.2. \( \square \)

### 6.2. Quasi-isometries.

**Theorem 6.2** ([Ger92]). Two of the eight 3-dimensional geometries, \( \widetilde{SL}_2(\mathbb{R}) \) and \( \mathbb{H}^2 \times \mathbb{R} \), are quasi-isometric.

Hence the unit tangent bundle \( U(S) \) of a closed surface \( S \) of genus greater than one and the product \( S \times S^1 \) have quasi-isometric fundamental groups. It follows that \( H_1 = \pi_1(U(S) \times S^1) \) and \( H_2 = \pi_1(S \times S^1 \times S^1) \) are quasi-isometric. Further note that \( H_1 \) is not commensurable with a Kähler group. This can be seen as follows: If there is a finite extension \( K \) of \( H_1 \) which is Kähler then clearly \( H_1 \) is Kähler. On the other hand, neither \( H_1 \) nor any finite index subgroup \( H \) of \( H_1 \) is Kähler: If \( X \) is the finite cover of \( U(S) \times S^1 \) corresponding to \( H \) then \( X = M \times S^1 \) where \( M \) is also a nontrivial circle bundle over a surface \( S' \). Now a presentation of \( \pi_1(M) \) is given by

\[
\langle a_1, b_1, \ldots, a_g, b_g, t \mid \prod_{i=1}^g [a_i, b_i] = t^k \rangle
\]

where \( \{a_1, b_1, \ldots, a_g, b_g\} \) is the usual set of generators for \( \pi_1(S') \) and \( k \) is a nonzero integer.
Abelianizing one sees that \( b_1(X) = b_1(S') + 1 \) and hence \( b_1(H) \) is odd. \( H_2 \) is clearly Kähler being the fundamental group of the product of a torus and an orientable 2-manifold. Thus we have an example demonstrating the fact that Kählerness is not preserved, even up to commensurability, by quasi-isometries within the category of compact aspherical complex surfaces. This gives a strong negative answer to a question of Gromov's (Problem on page 209 of [Gro93]).

6.3. Questions. One of the conclusions of Theorem 3.6 is that the quotient 3-manifold group \( Q \) could be the 3-dimensional Heisenberg group. It is not clear whether this situation can arise at all. Thus we ask

**Question 6.3.** Do there exist exact sequences

\[
1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1
\]

with \( G \) the 3-dimensional Heisenberg group, \( G \) a Kähler group and \( N \) finitely presented?

Question 6.3 seems to be related to the issue of finding which nilpotent groups may arise as Kähler groups. Campana [Cam95] gives examples of nilpotent Kähler groups \( H \) fitting into the exact sequence

\[
1 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow \mathbb{Z}^{\oplus 8} \longrightarrow 1
\]

with the presentation \( \{ t, x_1, y_1, \ldots, x_4, y_4 : [x_i, y_j] = \delta_{ij} t; [x_i, x_j] = 0 = [y_i, y_j] \} \). However, it is not clear which 2-step nilpotent groups may arise as Kähler groups. This seems to be the principal difficulty in addressing Question 6.3 at the moment.

We end with a simple observation based on Campana’s example above. It follows from the presentation of the group that the 3-dimensional Heisenberg group can in fact arise as a normal subgroup of a Kähler group. Let \( H_1 \subset H \) be the subgroup generated by \( \{ t, x_1, y_1 \} \). Then \( H_1 \) is isomorphic to the 3-dimensional Heisenberg group. Also the quotient group \( H/H_1 = \mathbb{Z}^{\oplus 6} \). Thus we may have non-Kähler finitely presented groups \( N \) (here the 3-dimensional Heisenberg group) appearing in an exact sequence

\[
1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1
\]

with both \( G \) and \( Q \) Kähler.

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THREE MANIFOLD GROUPS, KÄHLER GROUPS AND COMPLEX SURFACES


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