Evolutionary Games for Complex System

Rachid El-Azouzi

University of Avignon

Workshop on Social Networks
Bangalore, January 15th 2014

Joint work with: Nesrine Ben Khalifa, Yezekael Hayel and Issam Mabrouki
Outline

1. Game Theory
2. Evolutionary games
3. Limitation of existing Evolutionary games
4. Evolutionary Stable Strategies in Interacting communities
5. Replicator Dynamic
6. Conclusions and perspective
Game Theory

- A mathematical formalism for understanding, designing and predicting the outcome of games.

What is a game?
- Characterized by a number of players (2 or more), assumed to be intelligent and rational, that interact with each other by selecting various actions, based on their assigned preferences.

Players: decision makers
- A set of actions available for each player
- A set of preference relationships defined for each player for each possible action tuple.
  - Usually measured by the utility that a particular user gets from selecting that particular action intelligent and rational
## Classification of games

<table>
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<th>Non-cooperative</th>
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<td>Static</td>
<td>Dynamic (repeated)</td>
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<td>Strategic-form</td>
<td>Extensive-form</td>
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<tr>
<td>Perfect information</td>
<td>Imperfect information</td>
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<tr>
<td>Complete information</td>
<td>Incomplete information</td>
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- **Perfect information**: each player knows the identity of other players and, for each of them, the payoff resulting of each strategy.
- **Complete information**: each player can observe the action of each other player.
Nash Equilibrium

- Strategic game 3 basic components
  - A set of 2 or more players ($N = \{1, 2, \ldots, n\}$)
  - A set of actions for each player ($A_i$)
  - Utility function for every player ($U_i$)

- Nash Equilibrium: an action vector $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a Nash equilibrium if for all $i \in N$

$$U_i(s_i^*, s_{-i}^*) \geq U_i(s_i, s_{-i}^*) \quad \forall s_i$$

![Prisoner's dilemma matrix](image)
Evolutionary Games

- Is One-shot normal form game approach sufficient?
- How users can update their policies
  - implement their algorithms?
  - learn, correct their errors?
- Repeated game approach has been used.
  - same players,
  - the same one-shot game is played many times (depending on the state in Stochastic Games).
  - there are more strategies (mixed strategy, behavioral strategy, stationary strategy)
- In Game theory evolving:
  - a player cannot be faced to the same opponents,
  - there are many local interaction at the same time; characterization of all the system (with large number of users) at each time.
Evolutionary Strategy

- Introduced by Maynard Smith and Price (1973)
- There are two advantages in doing so within the framework of evolutionary games:
  - it provides the stronger concept of equilibria, the ESS, which allows us to identify robustness against deviations of more than one user, and
  - It is based on an evolutionary process, which is dynamic in nature which can model and capture the adaptation of agents to change their strategies and reach equilibrium over time
- Natural selection replaces rational behavior
- Evolutionary Strategy State (ESS):
  Suppose that the whole population uses a strategy $s$ and that a small fraction $\epsilon$ (called "mutations") adopts strategy $z$. $s$ is an ESS if for every $z \neq s$ there exists some $\epsilon_z > 0$ such that for all $\epsilon \in (0, \epsilon_z)$:

$$J(s, \epsilon z + (1 - \epsilon)s) > J(z, \epsilon z + (1 - \epsilon)s)$$
Solution concepts
Replicator dynamic

- Population can be divided into multiple groups, and each group adopts a different pure strategy.
- Replicator dynamics can model the evolution of the group size over time (unlike ESS, in replicator dynamics agents will play only pure strategies).
- The proportion or fraction of agents using pure strategy $a$ (i.e., population share) is denoted by $s_a(t)$ whose vector is $s(t)$.
- Let payoff of an agent using strategy $a$ given the population state $s$ be denoted by $U(a, s)$.
- Average payoff of the population, which is the payoff of an agent selected randomly from a population, is given by

$$\bar{U}(s(t)) = \sum_{a \in A} s_a U(a, s(t))$$
Replicator dynamic

- The reproduction rate of each agent (i.e., the rate at which the agent switches from one strategy to another) depends on the payoff (agents will switch to strategy that leads to higher payoff)
- Group size of agents ensuring higher payoff will grow over time because the agents having low payoff will switch their strategies
- Dynamics (time derivative) of the population share can be expressed as follows:
  \[
  \dot{s}_a(t) = s_a(t)(U(a, s(t)) - \bar{U}(s(t)))
  \]
- Evolutionary equilibrium can be determined at \( \dot{s}_a = 0 \)
Evolutionary games for complex system

- The EG has been focused only on uniform interactions between individuals groups:
- Realistic interactions are inherently non-uniform due to some barriers between agents as culture, language, spatial differences, etc.
  ⇒ any individual is more likely to meet and interact with some agents than others.
- It also assumed the payoff of a interaction is the same with different opponents.
- In many examples in social networks and biology systems, we observe that the population is composed into several communities or groups and the interaction is not uniform.
Multiple communities with non-uniform interaction

- A large population of players or individuals divided into \( N \) classes and competing through random pairwise interactions;
- The population profile is \( s = (s_1, \ldots, s_N) \);
- The payoff function \( U_i \) is given by
  \[
  U_i(k, s, p) = \sum_{j=1}^{N} p_{ij} e_k A_{ij} s_j,
  \]
  where \( e_k \) is the \( k \)-th element of the canonical basis of \( \mathbb{R}^n_i \) and \( A_{ij} \) is the payoff matrix;
- The expected payoff to an individual in community \( i \) is
  \[
  \bar{U}_i(s_i, s, p) = \sum_{k=1}^{n_i} s_{ik} U_i(k, s, p).
  \]
An *Evolutionary Stable Strategy* (ESS) is a state that, when adopted by an entire population, remains robust against a small fraction of mutants using a different strategy.

In multiple-community settings, different ESS characterizations can be inferred, which differ in the *level of stability*.

A strong ESS, is a strategy that, when adopted by the whole population, remains robust against invasion from a whole small fraction of mutants.

⇒ A state $s^*$ is a strong ESS, if for all $s \neq s^*$, there exists an $\epsilon(s) > 0$ such that for all $i = 1, \ldots, N$ and $\epsilon \leq \epsilon(s)$

$$
\bar{U}_i(s_i, \epsilon s + (1 - \epsilon)s^*, p) < \bar{U}_i(s^*_i, \epsilon s + (1 - \epsilon)s^*, p).
$$
A **weak ESS** is a strategy in which any community cannot be successfully invaded by a small fraction of local mutants from that community.

⇒ A state $s^*$ is a weak ESS if for all $s \neq s^*$ and for all $i = 1, \ldots, N$, there exists $\epsilon_i(s) > 0$ such that for all $\epsilon_i \leq \epsilon_i(s)$

$$
\bar{U}_i(s_i, (s_1^*, \ldots, \epsilon s_i + (1 - \epsilon) s_i^*, \ldots s_N^*), p) < \bar{U}_i(s_i^*, (s_1^*, \ldots, \epsilon s_i + (1 - \epsilon) s_i^*, \ldots s_N^*), p).
$$

An **intermediate ESS** is a strategy that when adopted by all the population, cannot be invaded by a whole small fraction of mutants, by considering the total fitness of the population:

⇒ A state $s^*$ is an intermediate ESS if for all $s \neq s^*$, there exists an $\epsilon(s) > 0$ such that for all $\epsilon \leq \epsilon(s)$

$$
\sum_{i \in \Gamma} \bar{U}_i(s_i, \epsilon s + (1 - \epsilon) s^*, p) < \sum_{i \in \Gamma} \bar{U}_i(s_i^*, \epsilon s + (1 - \epsilon) s^*, p).
$$

A strong ESS ⇒ An intermediate ESS ⇒ a weak ESS.
Two-communities and two-strategies

2-community 2-strategy settings

- Two large communities;
- The probability of intra-community interaction is $p$;
- **Pairwise** interactions described by the matrices of intra-community interaction $A, D$ and of inter-community interaction, $B, C$:

  $A = \begin{pmatrix} G_1 & H_1 \\ a_1 & b_1 & c_1 & d_1 \end{pmatrix}$, $D = \begin{pmatrix} G_2 & H_2 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$, $B = \begin{pmatrix} G_2 & H_2 \\ a & b & c & d \end{pmatrix}$, $C = B'$.

Existence of strong ESS

- If a strong ESS exists, it is necessary a dominate strategy
Mixed Nash Equilibrium

Mixed Nash equilibrium

Let

$$s_i^* = \frac{(1 - p)K_{-i}L - pK_iL_{-i}}{\Delta}, \ i = 1, 2.$$  

where $\Delta = p^2 L_1 L_2 - (1 - p)^2 L^2$, $L_i = (a_i - b_i - c_i + d_i)$, $L = a - b - c + d$ and $K_i = p(b_i - d_i) + (1 - p)(x_i - d)$, $x_1 = b$; $x_2 = c$.

$s^*$ is a mixed Nash equilibrium if

- $\Delta > 0$ and $(1 - p)K_{-i}L - pK_iL_{-i} > 0$, and $\Delta > (1 - p)K_{-i}L - pK_iL_{-i}$
- $\Delta < 0$ and $(1 - p)K_{-i}L - pK_iL_{-i} < 0$ and $\Delta < (1 - p)K_{-i}L - pK_iL_{-i}$
Intermediate ESS and Weak ESS

Intermediate ESS
The mixed Nash equilibrium is an intermediate ESS if and only if $L_1 < 0$ and $\Delta > 0$.

Weak ESS
The mixed Nash equilibrium is a weak ESS if and only if $L_1 < 0$ and $L_2 < 0$. 
Replicator Dynamic

The replicator dynamic is a commonly used tool to observe the asymptotic dynamic of strategy changes in an evolutionary process. Its equations write

\[
\begin{align*}
\dot{s}_1(t) &= s_1(t)(1 - s_1(t)) \left[ p s_1(t) L_1 + (1 - p) s_2(t) L + K_1 \right], \\
\dot{s}_2(t) &= s_2(t)(1 - s_2(t)) \left[ p s_2(t) L_2 + (1 - p) s_1(t) L + K_2 \right];
\end{align*}
\]

with an interior stationary point \( s^* = (s_1^*, s_2^*) \), \( s_i^* = \frac{(1 - p)K_{-i}L - pK_iL_{-i}}{\Delta} \).

- If \( L_1 < 0 \) and \( \Delta = p^2L_1L_2 - (1 - p)^2L^2 > 0 \), then \( s^* \) is globally asymptotically stable for the replicator dynamic.
- Any mixed intermediate ESS is globally asymptotically stable for the replicator dynamic.
Application to Prisoner's Dilemma - Example 1

- Two agents choose a strategy of cooperate or defect
- Payoff matrices
  \[ A_1 = \begin{pmatrix} C & D^* \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 13 & 4 \end{pmatrix}, \quad D_1 = \begin{pmatrix} C & D^* \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 15 & 1 \end{pmatrix}, \quad B_1 = C_1 = \begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} 7 & 13 \\ 9 & 1 \end{pmatrix}. \]

* dominant strategy

Fig. Mixed intermediate ESS
Application to Prisoner’s Dilemma - Example 1

Fig. Replicator Dynamic for two different probability interactions. (a) $s^* = (0.27, 0.08)$ is the mixed intermediate ESS. (b) C is the dominant strategy in both communities.
Application to Prisoner’s Dilemma - Example 2

\[ A_1 = \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 13 & 4 \end{pmatrix}, \quad D_1 = \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 15 & 1 \end{pmatrix}, \quad B_2 = C_2 = \begin{pmatrix} C^* & D \end{pmatrix} \begin{pmatrix} 11 & 5 \\ 4 & 0 \end{pmatrix}. \]

* dominant strategy

Fig. Mixed intermediate ESS
Numerical Examples

Application to Prisoner’s Dilemma - Example 2

(a) $p = 0.3$

(b) $p = 0.57$

Fig. **Replicator Dynamic** for two different probability interactions. (a) C is the dominant strategy in both communities. (b) $s^* = (0.42, 0.53)$ is the mixed intermediate ESS.
Delayed Replicator Dynamic

- **Strategic delay**: delay represents the time between the use of a strategy and the time the user feels the impact of his strategy.

- **Spatial delay**: delays are not associated with the strategy used by an individual, but rather to the opponent with which an individual interacts.

- **Strategic-Spatial delay**: delay is associated with both the strategy and the opponent with which an individual interacts.

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1 Hamidou Tembine, Eitan Altman and Rachid Elazouzi "Asymmetric Delay in Evolutionary Games", in the proceeding of Valuetools, Nantes, France, 2007
Symmetric Strategic Delay

Replicator dynamics (RD)

\[ \dot{s}_i(t) = s_i(t)(1 - s_i(t))\left(U_1(G_1, s(t - \tau), p) - U_1(H_1, s(t - \tau), p)\right) \]

where \( s = (s_1, s_2) \)

The solution of the RD is asymptotically stable if all roots of the characteristic equation have negative real parts

Characteristic equation

\[ \lambda^2 - \lambda \left[p_2 \gamma_2 L_2 + p_1 \gamma_1 L_1\right] e^{-\lambda \tau} + \gamma_1 \gamma_2 p_1 p_2 \left[L_1 L_2 - L^2\right] e^{-2\lambda \tau} = 0. \]

Result

\( s^* \) is asymptotically stable if and only if \( \tau < \max\left(\frac{\pi}{2|\lambda_+|}, \frac{\pi}{2|\lambda_-|}\right) \), with \( \lambda_{\pm} = \frac{p_1 \gamma_1 L_1 + p_2 \gamma_2 L_2 \pm \sqrt{\beta}}{2}, \gamma_i = s_i^*(1 - s_i^*) \) and

\[ \beta = [p_1 \gamma_1 L_1 + p_2 \gamma_2 L_2]^2 - 4\gamma_1 \gamma_2 p_1 p_2 [L_1 L_2 - L^2]. \]
Spatial delay

Replicator dynamics (RD)

\[
\dot{s}_i(t) = s_i(t)(1 - s_i(t))\left(U_i(G_i, s(t, \tau), p) - U_i(H_i, s(t, \tau), p)\right)
\]

where \(s(t, \tau) = (s_i(t), s_{-i}(t - \tau))\).

Result

An intermediate ESS is asymptotically stable for any delay \(\tau\).
Fig. Replicator Dynamic (a) with symmetric strategic delay. (b) with symmetric spatial delay.
Fig. Replicator Dynamic with strategic-spatial delay
Conclusions and Further Work

(i) In two-community two-strategy model any mixed (interior) Nash equilibrium cannot be a strong ESS.

(ii) We showed that the mixed Nash equilibrium may be an intermediate ESS or a Weak ESS.
- The Intermediate ESS is asymptotically stable for the replicator dynamic.
- The ESS can be unstable for large strategy delay but it remains stable for any spatial delay
- Perspectives:
  - More complex topology
  - More than two players interactions
  - Group Equilibrium Stable Strategy: This new concept allows to model the evolution of population by taking into account the collective behavior