The Volume of the Caracol Polytope

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Overview

- We present a new method for computing volumes of flow polytopes by giving a combinatorial interpretation of the Lidskii volume formula through objects called unified diagrams.
- We use our method to show that the volume of the caracol polytope is the product of a Catalan number and the number of parking functions.

Flow polytopes

Let G be an acyclic directed graph with $n+1$ vertices and m edges. Given a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, an $\mathbf{a}$-flow on G is a tuple $(b_{ij})_{(i,j)\in E(G)}$ of real numbers such that for $j = 1, \ldots, n$,

$$\sum_{(j,k)\in E(G)} b_{jk} - \sum_{(i,j)\in E(G)} b_{ij} = a_j.$$

We view an $\mathbf{a}$-flow on G as an assignment of flow b_{ij} to the edge (i,j) such that the net flow at vertex j is a_j. The set $\mathcal{F}_G(\mathbf{a})$ of $\mathbf{a}$-flows of G can be viewed as a polytope in $\mathbb{R}^m$ and it is called the flow polytope of G with net flow $\mathbf{a}$.

The Kostant partition function

By associating the vector $\mathbf{e}_i - \mathbf{e}_j$ to the edge (i,j), an $\mathbf{a}$-flow on G is equivalent to the expression of the vector $\mathbf{a}' = \sum_{(i,j)\in E(G)} b_{ij}[\mathbf{e}_i - \mathbf{e}_j]$ as a linear combination of the positive roots in the set

$$\Phi_G^+ = \{\mathbf{e}_i - \mathbf{e}_j \mid (i,j) \in E(G)\}.$$

The number of integral $\mathbf{a}$-flows on G is called the Kostant partition function of G evaluated at $\mathbf{a}'$ and we denote it by $K_G(\mathbf{a}')$. This enumerates the number of lattice points of $\mathcal{F}_G(\mathbf{a})$.

The Lidskii volume formula

A remarkable formula for the normalized volume of a flow polytope was obtained by Baldoni and Vergne using residue techniques [1]. This formula was also proved by Mészáros and Morales using polytope subdivisions [3].

Lidskii formula. Let G be a directed graph with $n+1$ vertices and m edges. Let $\mathbf{t} = (t_1, \ldots, t_n)$ be the shifted out-degree vector whose i-th entry is one less than the out-degree of vertex i. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, and let $G|_n$ denote the restriction of G to its first n vertices. Then

$$\operatorname{vol} \mathcal{F}_G(\mathbf{a}) = \sum_{\mathbf{s} \rhd \mathbf{t}} \binom{m-n}{\mathbf{s}} \cdot \mathbf{a}^{\mathbf{s}} \cdot K_{G|_n}(\mathbf{s} - \mathbf{t}),$$

where the sum is over weak compositions $\mathbf{s} = (s_1, \ldots, s_n) \vDash m-n$ that dominate $\mathbf{t}$.

Corollary. The special case when $\mathbf{a} = (1, 0, \ldots, 0)$ is

$$\operatorname{vol} \mathcal{F}_G(1, 0, \ldots, 0) = K_{G|_n}(m - n - t_1, -t_2, \ldots, -t_n).$$

Motivation

For certain graphs G and net flow vectors $\mathbf{a}$, the volume of $\mathcal{F}_G(\mathbf{a})$ has a nice combinatorial formula. We highlight a few examples which are pertinent to our work.

- When G is the complete graph K_{n+1} and $\mathbf{a} = (1, 0, \ldots, 0)$, $\mathcal{F}_G(\mathbf{a})$ is the Chan–Robbins–Yuen polytope, and in the case $\mathbf{a} = (1, \ldots, 1)$, $\mathcal{F}_G(\mathbf{a})$ is the Tesler polytope. We have
$$\operatorname{vol} \mathrm{CRY}_{n+1} = \prod_{i=1}^{n-2} C_i, \quad \text{and} \quad \operatorname{vol} \mathrm{Tes}_{n+1} = \frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}} \prod_{i=1}^{n-1} C_i,$$
where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number. The only known proofs of these use a variant of the Morris constant term identity.
- When G is the Pitman–Stanley graph PS_{n+1}, $\mathcal{F}_{\mathrm{PS}_{n+1}}(1, \ldots, 1)$ is affinely equivalent to the Pitman–Stanley polytope and
$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+1}}(1, \ldots, 1) = n^{n-2}.$$
- When G is the caracol graph Car_{n+1}, $\mathcal{F}_{\mathrm{Car}_{n+1}}(1, 0, \ldots, 0)$ is equivalent to the order polytope of the poset $[2] \times [n-2]$, and
$$\operatorname{vol} \mathcal{F}_{\mathrm{Car}_{n+1}}(1, 0, \ldots, 0) = C_{n-2}.$$

References

[1] Welleda Baldoni and Michèle Vergne. "Kostant partition functions and flow polytopes". In: *Transform. Groups* 13.3-4 (2008), pp. 447–469.

[2] Carolina Benedetti et al. "A combinatorial model for computing volumes of flow polytopes". URL: https://arxiv.org/pdf/1801.07684.

[3] Karola Mészáros and Alejandro H. Morales. "Volumes and Ehrhart polynomials of flow polytopes". URL: https://arxiv.org/pdf/1710.00701.

This project was initiated at the Polyhedral Geometry and Partition Theory workshop at the American Institute of Mathematics. We are grateful to the organizers of the workshop - Federico Ardila, Benjamin Braun, Peter Paule, and Carla Savage - and to AIM which led to this collaboration.

Gravity diagrams

Let $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 1, \ldots, n$. A line-dot diagram is a pictorial representation of a vector partition of the vector

$$\mathbf{c}' = \sum_{i=1}^{n+1} c_i \mathbf{e}_i = \sum_{i=1}^{n} (c_1 + \cdots + c_i)\alpha_i$$

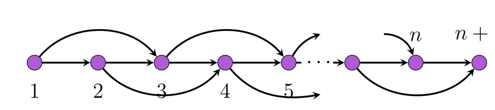
into the positive roots of Φ_G^+. It consists of an array of $c_1 + \cdots + c_i$ dots in the i-th column, and a part $[\mathbf{e}_i - \mathbf{e}_j]$ of the vector partition is represented by a line through dots in the i-th to the j-th column.

Two line-dots diagrams are equivalent if they represent the same vector partition. A gravity diagram is an equivalence class of line-dot diagrams, and we let $\mathrm{GD}_G(\mathbf{c}')$ denote the set of classes. The choice of a class representative depends on the graph G.

Theorem 1. The Kostant partition function of the graph G evaluated at $\mathbf{c}'$ is the number of gravity diagrams.

$$K_G(\mathbf{c}') = |\,\mathrm{GD}_G(\mathbf{c}')|.$$

The zigzag graph. The graph Zig_{n+1} has $m = 2n - 1$ edges



and shifted out-degree vector $\mathbf{t} = (1, \ldots, 1, 0)$. By the Corollary to the Lidskii formula and Theorem 1,

$$\operatorname{vol} \mathcal{F}_{\mathrm{Zig}_{n+1}}(1, 0, \ldots, 0) = K_{\mathrm{Zig}_n}(\mathbf{c}') = |\,\mathrm{GD}_{\mathrm{Zig}_n}(\mathbf{c}')|,$$

where $\mathbf{c}' = \sum_{i=1}^{n-2}(n-1-i)\alpha_i$. Since

$$\Phi^+_{\mathrm{Zig}_n} = \{\alpha_1, \ldots, \alpha_{n-1}\} \cup \{\alpha_i + \alpha_{i+1} \mid i = 1, \ldots, n-2\},$$

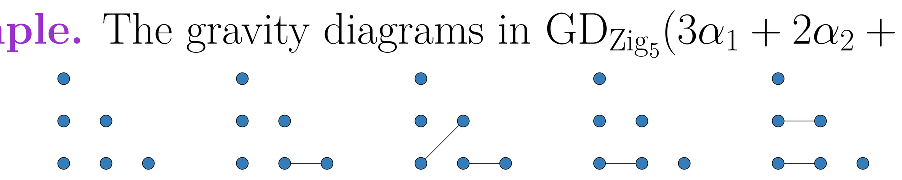
then a gravity diagram in $\mathrm{GD}_{\mathrm{Zig}_n}(\mathbf{c}')$ is a triangular array of $n-2$ columns of dots whose lines may only connect dots in two consecutive columns, and by our convention the diagram is constructed by placing lines from right to left such that each line occupies the lowest available dots in their respective columns. We enumerate these diagrams to obtain the next Proposition.

Proposition 2. The volume of the zigzag polytope is

$$\operatorname{vol} \mathcal{F}_{\mathrm{Zig}_{n+1}}(1, 0, \ldots, 0) = E_{n-1},$$

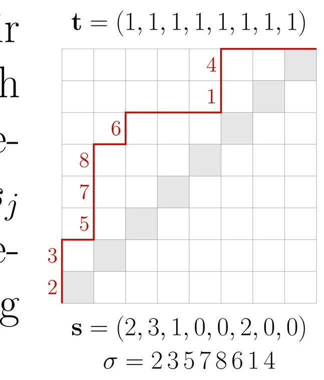
where the Euler number E_{n-1} is the number of alternating permutations on $n-1$ letters.

Example. The gravity diagrams in $\mathrm{GD}_{\mathrm{Zig}_5}(3\alpha_1 + 2\alpha_2 + \alpha_3)$ are



Labeled t-Dyck paths

Given $\mathbf{t} \in \mathbb{Z}^n_{\geq 0}$, a labeled $\mathbf{t}$-Dyck path is a pair $(\mathbf{s}, \sigma)$ where $\mathbf{s}$ is a weak composition of $|\mathbf{t}|$ such that $\mathbf{s} \rhd \mathbf{t}$, and σ is a permutation of $|\mathbf{t}|$ whose descents can possibly occur in positions $s_1 + \cdots + s_j$ for $j = 1, \ldots, |\mathbf{t}| - 1$. The figure to the right depicts a $(1, \ldots, 1)$-Dyck path, which is a parking function in the classical sense.



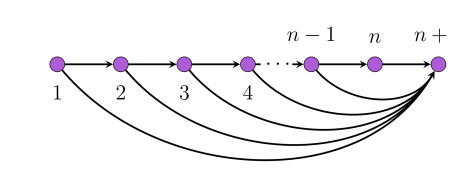
Unified diagrams

A unified diagram for the flow polytope $\mathcal{F}_G(\mathbf{a})$ with shifted out-degree vector $\mathbf{t}$ is a tuple $U = (\mathbf{s}, \sigma, \varphi, D)$ where $(\mathbf{s}, \sigma)$ is a labeled $\mathbf{t}$-Dyck path, $\varphi \in [a_1]^{s_1} \times \cdots \times [a_n]^{s_n}$, and D is a gravity diagram in $\mathrm{GD}_{G|_n}(\mathbf{s} - \mathbf{t})$. We let $\mathrm{U}_G(\mathbf{a})$ denote the set of unified diagrams.

Theorem 3. The volume of the flow polytope $\mathcal{F}_G(\mathbf{a})$ is the number of unified diagrams.

$$\operatorname{vol} \mathcal{F}_G(\mathbf{a}) = |\,\mathrm{U}_G(\mathbf{a})|.$$

The Pitman–Stanley graph. The graph PS_{n+1}



has $m = 2n - 1$ edges and shifted out-degree vector $\mathbf{t} = (1, \ldots, 1, 0)$. The restriction $G|_n$ is simply the path on n vertices so there is a unique gravity diagram for every Dyck path $\mathbf{s} \rhd \mathbf{t}$. As such, a unified diagram in $\mathrm{U}_{\mathrm{PS}_{n+1}}(1, \ldots, 1)$ is completely characterized by its labeled $\mathbf{t}$-Dyck path, which can be identified with a parking function.

Proposition 4. The volume of the Pitman–Stanley polytope is the number of parking functions

$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+1}}(1, \ldots, 1) = n^{n-2}.$$

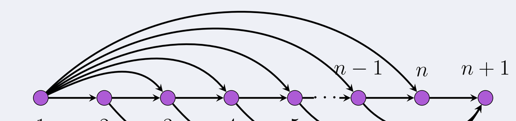
For $i \geq 0$, the level-i unified diagrams $\mathrm{U}^i_G(\mathbf{a})$ is the set of unified diagrams whose first column north steps are omitted and whose first east step begins at the i-th line.

Theorem 5. We use the following refined formula to compute the volume of the caracol polytope in the next section.

$$\operatorname{vol} \mathcal{F}_G(\mathbf{a}) = |\,\mathrm{U}_G(\mathbf{a})| = \sum_{i=0}^{|\mathbf{t}|} \binom{|\mathbf{t}|}{i} a_1^{|\mathbf{t}|-i} |\,\mathrm{U}^i_G(\mathbf{a})|.$$

The caracol polytope

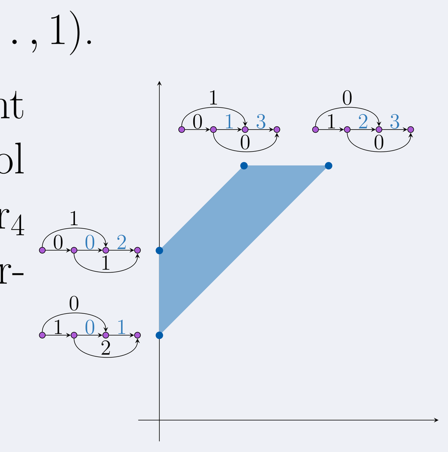
The caracol graph. The graph Car_{n+1} has $m = 3n - 4$ edges and shifted out-degree vector $\mathbf{t} = (n-2, 1, \ldots, 1, 0)$.



The caracol polytope is $\mathcal{F}_{\mathrm{Car}_{n+1}}(1, \ldots, 1)$.

Example. The figure to the right depicts the two-dimensional caracol polytope of flows on the graph Car_4 with net flow $\mathbf{a} = (1, 1, 1)$. Its normalized volume is

$$\operatorname{vol} \mathcal{F}_{\mathrm{Car}_4}(1, 1, 1) = C_1 \cdot 3^1 = 3.$$

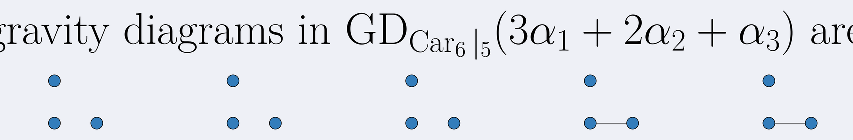


Gravity diagrams. Let $\mathbf{c}' = \sum_{i=1}^{n-2}(n-1-i)\alpha_i$. Since

$$\Phi^+_{\mathrm{Car}_{n+1}|_n} = \{\alpha_1, \ldots, \alpha_{n-1}\} \cup \{\alpha_1 + \cdots + \alpha_i \mid i = 2, \ldots, n-1\},$$

a gravity diagram in $\mathrm{GD}_{\mathrm{Car}_{n+1}|_n}(\mathbf{c}')$ is a triangular array of $n-2$ columns of dots whose lines are horizontal and left-justified, and by our convention the longer lines are bottom-justified.

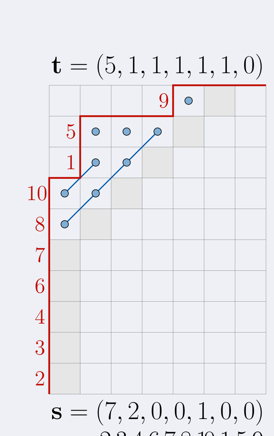
The gravity diagrams in $\mathrm{GD}_{\mathrm{Car}_6|_5}(3\alpha_1 + 2\alpha_2 + \alpha_3)$ are



Proposition 6. The volume of the flow polytope of Car_{n+1} with net flow $(1, 0, \ldots, 0)$ is the Catalan number

$$\operatorname{vol} \mathcal{F}_{\mathrm{Car}_{n+1}}(1, 0, \ldots, 0) = C_{n-2}.$$

Unified diagrams. The figure to the right depicts a unified diagram $U = (\mathbf{s}, \sigma, D)$ for $\mathcal{F}_{\mathrm{Car}_8}(1, \ldots, 1)$ with shifted out-degree vector $\mathbf{t}$. The $\mathbf{t}$-Dyck path is $\mathbf{s}$ with parking label σ, and D is a gravity diagram in $\mathrm{GD}_{\mathrm{Car}_8|_7}(2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5)$. When $\mathbf{a} = (1, \ldots, 1)$, the net flow label φ may be suppressed.



The parking triangle.

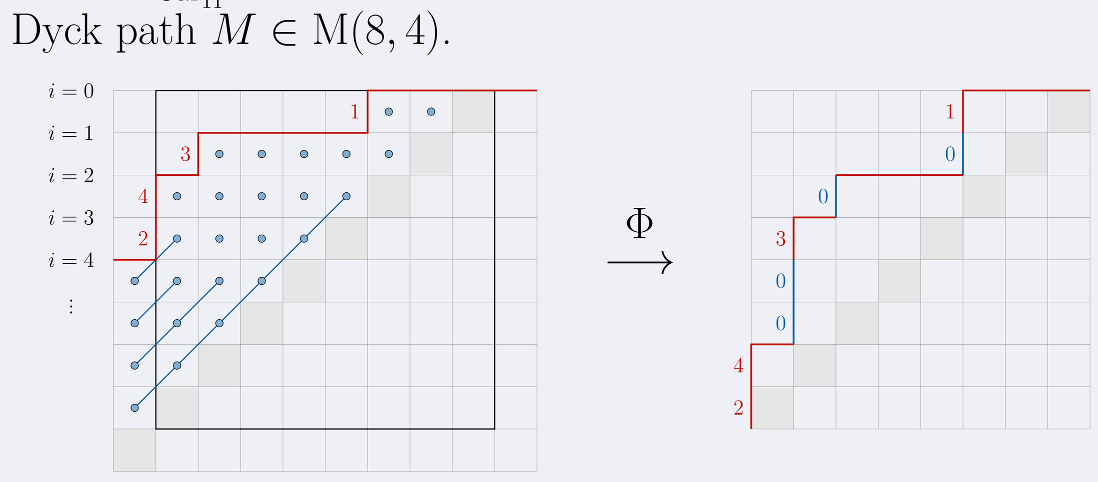
1
1 1
2 3 3
5 10 16 16
14 35 75 125 125
42 126 336 756 1296 1296

For $r \geq 0$, the numbers in the r-th row of the parking triangle enumerate the level-i unified diagrams for the caracol polytope $\mathcal{F}_{\mathrm{Car}_{r+3}}(1, \ldots, 1)$. The numbers along the r-th row interpolate between the Catalan number C_r and the number $(r+1)^{r-1}$ of parking functions of r.

Theorem 7. The number of level-i unified diagrams for the caracol polytope $\mathcal{F}_{\mathrm{Car}_{n+1}}(1, \ldots, 1)$ is

$$|\,\mathrm{U}^i_{\mathrm{Car}_{n+1}}(1, \ldots, 1)| = (n-1)^{i-1}\binom{2(n-1)-i}{n-1}.$$

Example. The proof of Theorem 7 is obtained by constructing a bijection Φ from the set of level-i unified diagrams $\mathrm{U}_{\mathrm{Car}_{n+1}}(1, \ldots, 1)$ to the set $\mathrm{M}(n-2, i)$ of Dyck paths from $(0,0)$ to $(n-2, n-2)$ which are labeled by the multiset $\{0^{n-2-i}, 1, \ldots, i\}$. Below is a level-4 unified diagram $U \in \mathrm{U}_{\mathrm{Car}_{11}}^4(1, \ldots, 1)$ and its corresponding multiset-labeled Dyck path $M \in \mathrm{M}(8, 4)$.



Our final result follows by combining Theorems 5 and 7.

Theorem 8. The volume of the caracol polytope is the product of a Catalan number and the number of parking functions.

$$\operatorname{vol} \mathcal{F}_{\mathrm{Car}_{n+1}}(1, \ldots, 1) = C_{n-2} \cdot n^{n-2}.$$

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