Generalized nil-Coxeter algebras

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Abstract

- Nil-Coxeter algebras are associated graded algebras of real reflection groups.
- Motivated by complex reflection groups – and a problem of Coxeter (1957) – we define *generalized* nil-Coxeter algebras, where
  (a) the braid relations continue to hold; and
  (b) the nilpotence degree is allowed to vary.
- We first obtain a new finite-dimensional 'Type A' family of such algebras – called $NC_A(n,d)$ – and study its properties.
- We then classify all finite-dimensional generalized nil-Coxeter algebras:
  (a) These include the usual nil-Coxeter algebras over finite Coxeter groups.
  (b) The *only* other finite-dimensional cases are the family $NC_A(n,d)$.

Usual and generalized nil-Coxeter algebras

$\Bbbk$ is any unital commutative ground ring.

Nil-Coxeter algebras over real reflection groups

Definition 1. Given a finite index set I, a *Coxeter matrix* is $M_{I\times I}$ such that $m_{ij} = m_{ji} \in \mathbb{Z}^{\geqslant 0} \sqcup \{\infty\}$ for $i \neq j$. The corresponding braid monoid $\mathcal{B}_M$ has generators T_i, $i \in I$, and relations

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij}\ times} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}\ times}, \qquad \forall i \neq j \in I.$$

This defines three $\Bbbk$-algebras:

1. The *Coxeter group algebra* $\Bbbk W(M)$, where $W(M)$ is the real reflection group $\mathcal{B}_M/(T_i^2 - 1,\ \forall i)$.
2. The 0*-Hecke algebra* $\Bbbk\mathcal{B}_M/(T_i^2 - T_i,\ \forall i)$.
3. The nil-Coxeter algebra $NC_M := \Bbbk\mathcal{B}_M/(T_i^2,\ \forall i)$.

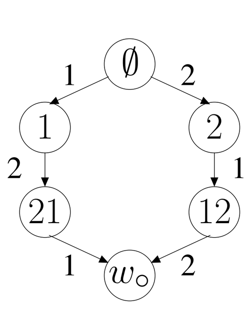

Figure 1: The 'usual' Type A_2 nil-Coxeter algebra

Generic Hecke algebras encompass all of these families:

$$\mathcal{E}_M := \Bbbk\mathcal{B}_M/(T_i^2 - a_i T_i - b_i),$$

(where a_i, b_i are scalars), and the associated graded algebra is NC_M.

Fact: $\mathcal{E}_M$ is a *flat deformation* of NC_M:

$$\dim \mathcal{E}_M = \dim NC_M = |W(M)|.$$

. . . and over complex reflection groups

To every complex reflection group W are associated:

1. a generalized braid group/monoid $\mathcal{B}_W$;
2. a presentation 'a la Coxeter': *braid relations* and *order* relations $T_i^{d_i} = 1,\ i \in I$.
3. generic Hecke algebras $\mathcal{E}_W$.

What are the associated graded algebras of these algebras?

Definition 2. Suppose W is:

- a (finite or infinite) discrete real/complex reflection group W,
- with a finite generating set of real/complex reflections $\{T_i : i \in I\}$.

Given an integer tuple $\mathbf{d} := \{d_i \geqslant 2,\ \forall i \in I\}$, define the generalized nil-Coxeter algebra (for this data) to be

$$NC_W(\mathbf{d}) \ := \ \Bbbk\langle T_i : i \in I\rangle \ / \ (\text{ braid relns; } \ T_i^{d_i} = 0,\ \forall i\ ).$$

Questions: Suppose W is a finite complex reflection group, and $\Bbbk$ is a field of characteristic zero. Then $\dim \mathcal{E}_W = |W|$ by the Broué–Malle–Rouquier Freeness Conjecture.

1. Are generic Hecke algebras $\mathcal{E}_W$ (or the group algebra $\Bbbk W$) flat deformations of $NC_W(\mathbf{d})$ for suitable $\mathbf{d}$? (Or for *any* $\mathbf{d}$?)
2. Taking a step back: Are the algebras $NC_W(\mathbf{d})$ finite-dimensional for some (any) $\mathbf{d}$? (Marin recently obtained some negative results.)

Main question

More generally (than above): Classify the real/complex reflection groups W and integer vectors $\mathbf{d}$ such that $NC_W(\mathbf{d})$ is finite-dimensional.

1. *Real groups:* Only known examples: 'usual' nil-Coxeter algebras $NC_W((2,2,\dots,2))$.
2. *Complex groups:* No examples known.

A new finite-dimensional (type A) family: $NC_A(n,d)$

We first study generalized nil-Coxeter algebras of type A, motivated by the following classical work of Coxeter:

Let $W_{n,d}$ be the quotient of the Artin braid group on $n-1$ generators, by $T_i^d = 1$ for all $i = 1,\dots,n-1$. When is $W_{n,d}$ a finite group?

Theorem 3 (Coxeter, *Proc. Canad. Math. Cong.*, 1957)

The 'generalized Coxeter group' $W_{n,d}$ is finite if and only if $\frac{1}{n} + \frac{1}{d} > \frac{1}{2}$, in which case the group has size $\left(\frac{1}{n} + \frac{1}{d} - \frac{1}{2}\right)^{1-n} \frac{n!}{n^{n-1}}$.

Note that the corresponding generalized nil-Coxeter algebras $NC_{S_n}((d,d,\dots,d))$ are *not* finite-dimensional, unless $d = 2$ and we have the 'usual' Type A nil-Coxeter algebras.

The following 'correctly' parallels Coxeter's construction, and is the first 'non-usual' example of a finite-dimensional nil-Coxeter algebra:

Theorem 10 (Khare, [1], 2018)

For integers $n \geqslant 1$ and $d \geqslant 2$, define the $\Bbbk$-algebra

$$NC_A(n,d) := NC_{S_{n+1}}((2,\dots,2,d)). \tag{5}$$

Thus, $NC_A(n,d)$ has generators $T_1,\dots,T_n$, with relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad \forall\, 0 < i < n; \tag{6}$$
$$T_i T_j = T_j T_i, \qquad \forall\, |i-j| > 1; \tag{7}$$
$$T_1^2 = \cdots = T_{n-1}^2 = T_n^d = 0. \tag{8}$$

Moreover, $NC_A(n,d)$ has a Coxeter word basis of $n!(1 + n(d-1))$ generators

$$T_w,\ w \in S_n,$$
$$T_w T_n^k T_{n-1} T_{n-2} \cdots T_{m+1} T_m,\ w \in S_n,\ k \in [1, d-1],\ m \in [1, n].$$

Example 9. $NC_A(1,d) = \Bbbk[T_1]/(T_1^d)$, while $NC_A(2,d)$ is below:

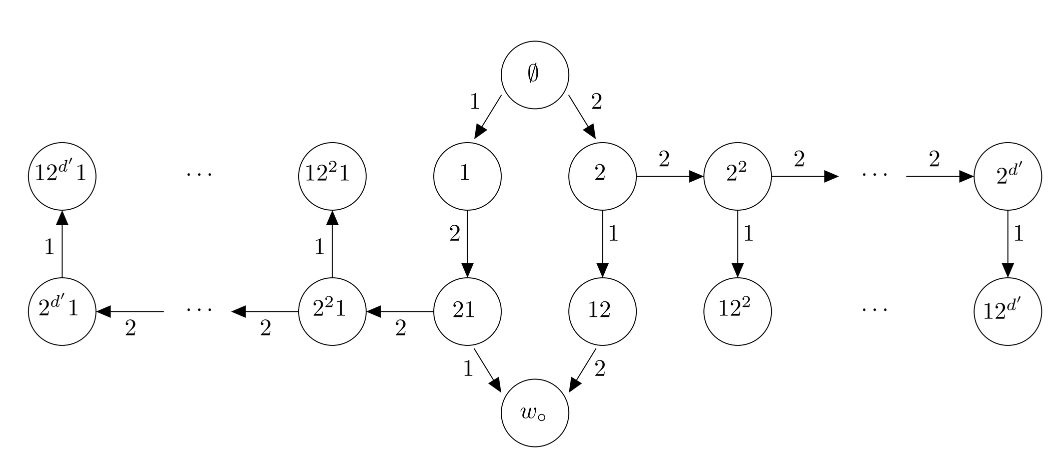

Figure 2: The nil-Coxeter algebra $NC_A(2,d)$, with $d' = d-1$

Further properties of $NC_A(n,d)$

The algebras $NC_A(n,d)$ resemble the 'usual' Type A nil-Coxeter algebras in several ways:

Theorem 9 (Khare, [1], 2018)

Fix integers $n \geqslant 1$ and $d \geqslant 2$.

1. The algebra $NC_A(n,d)$ has a length function that restricts to the usual length function $\ell_{A_{n-1}}$ on the 'usual nil-Coxeter' subalgebra generated by $T_1,\dots,T_{n-1}$; and
$$\ell(T_w T_n^k T_{n-1} \cdots T_m) = \ell_{A_{n-1}}(w) + k + n - m, \tag{10}$$
for all $w \in S_n$, $k \in [1, d-1]$, and $m \in [1, n]$.
2. There is a unique longest word $T_{w'_\circ} T_n^{d-1} T_{n-1} \cdots T_1$ of length
$$l_{n,d} := \ell_{A_{n-1}}(w'_\circ) + d + n - 2.$$
3. The algebra $NC_A(n,d)$ is local, with unique maximal (augmentation) ideal $\mathfrak{m}$ generated by $T_1,\dots,T_n$. Moreover, $\mathfrak{m}^{1+l_{n,d}} = 0$.

Thus there is a variant of the Coxeter word length, as well as a unique longest word. As an immediate consequence, one computes the Hilbert polynomial of the graded algebra $NC_A(n,d)$:

Corollary 11 (Khare, [1], 2018)

If $T_1,\dots,T_n$ all have degree 1, then $NC_A(n,d)$ has Hilbert–Poincaré series

$$[n]_q!\ (1 + [n]_q\, [d-1]_q), \qquad \text{where } [n]_q := \frac{q^n - 1}{q - 1},\ [n]_q! := \prod_{j=1}^n [j]_q.$$

Classification of finite-dimensional generalized nil-Coxeter algebras

Classification results in Coxeter-type settings are ubiquitous:

- Weyl, Coxeter, and complex reflection groups;
- Finite type quivers;
- McKay–Slodowy correspondence;
- simple Lie algebras;
- finite-dimensional Nichols algebras / pointed Hopf algebras;
- finite "generalized Coxeter groups" (Coxeter 1957; Koster PhD thesis 1975). . .

The next result classifies all finite-dimensional generalized nil-Coxeter algebras: the first 'non-usual' examples $NC_A(n,d)$ are the *only* ones!

Theorem 12 (Khare, [1], 2018)

Suppose W is any irreducible discrete real or complex reflection group (finite or infinite), and $\mathbf{d} \in (\mathbb{Z}_{\geqslant 2})^I$ is any integer vector. Then $NC_W(\mathbf{d})$ is finite-dimensional (i.e., a finitely generated $\Bbbk$-module) if and only if:

1. either W is a finite Coxeter group and $d_i = 2\ \forall i$ (the 'usual' nil-Coxeter algebras);
2. or W is of type A_n and $\mathbf{d} = (2,\dots,2,d)$ or $(d,2,\dots,2)$ for some $d > 2$. In other words, $NC_W(\mathbf{d}) = NC_A(n,d)$.

This shows a statement by Marin (2014):

A key difference between real and complex reflection groups W seems to be the lack of nil-Coxeter algebras of dimension precisely $|W|$ for the latter.

(Verified by Marin for a few cases.)

Further questions

1. Do the algebras $NC_A(n,d)$ for $d > 2$ occur as (differential) operators, e.g. on some polynomial ring? (For $d = 2$ their representation as divided difference operators is used to define Schubert polynomials.)
2. Type-free proof of the above Classification?
3. Categorify the 'usual' nil-Coxeter algebras? How about the algebras $NC_A(n,d)$? (First try $d = 1$.)

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Reference

[1] A. Khare, *Generalized nil-Coxeter algebras over complex reflection groups*, Trans. Amer. Math. Soc. 370(4), pages 2971–2999, *2018*.