E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)

	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	>90	76 — 90	61 — 75	46 — 60	35 — 45	<35

	Grade A$^+$	Grade A	Grade B$^+$	Grade B	Grade C	Grade D	Grade F
Marks-Range	>90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	<40

1. Basic Algebraic Concepts

Submit a solution of the $*$-Exercise ONLY. **Due Date :** Wednesday, 10-08-2011 (Before the Class)

1.1 (a) Let $G \subseteq \mathbb{Z}$ be a subset of integers which contains at least one positive integer and at least one negative integer. Suppose that G is closed under the usual addition in $\mathbb{Z}$ i.e. $a+b \in G$ whenever $a,b \in G$. Prove that $(G,+)$ is a group. (**Hint :** Use the minimum principle, see Supplement S1.1)

(b) Let $N \subseteq \mathbb{N}$ be a submonoid $\neq 0$ of the additive monoid $\mathbb{N}$ with the following property : If $a,b \in N$ and $a \leq b$, then $b-a \in N$. Show that $N = \mathbb{N}n = \{an \mid a \in \mathbb{N}\}$ for some uniquely determined $n \in N^* := N \cap \mathbb{N}^*$. (**Remark :** For application see Supplement S1.4.)

1.2 For $a,b \in \mathbb{R}$, let $f_{a,b} : \mathbb{R} \to \mathbb{R}$ be the function defined by $f_{a,b}(x) := ax+b$, $x \in \mathbb{R}$. Then $\operatorname{Aff}(1,\mathbb{R}) := \{f_{a,b} \mid a,b \in \mathbb{R},\, a \neq 0\}$ with the composition as a binary operation is not a commutative group. (**Remark :** The group $\operatorname{Aff}(1,\mathbb{R})$ is called the affine group of $\mathbb{R}$ and its elements are called the affine linear maps.)

1.3 A non-empty *finite* subset H of a group G is a subgroup of G if $ab \in H$ whenever $a,b \in H$. A *finite* submonoid H of a group G is a subgroup of G. (**Hint :** If $a \in H$, then left translation $\lambda_a : H \to H$ is injective and hence bijective by the Pigeonhole Principle[1])

***1.4 (a)** Let G be a finite group with the identity element e. Suppose that $\#G = n$ and $(a_1,\ldots,a_n) \in G^n = G \times \cdots \times G$ (n-times). Then there exist r,s with $0 \leq r < s \leq n$ such that $a_{r+1}\cdots a_s = e$. (**Hint :** The $n+1$ products $a_1 \cdots a_s$, $s=0,\ldots,n$, cannot be pairwise distinct. Note that $a_1 \cdots a_s := e$ for $s=0$.)

(b) For any given $a_1,\ldots,a_n \in \mathbb{Z}$, $n \in \mathbb{N}^+$, show that there exist r,s with $0 \leq r < s \leq n$ such that $a_{r+1} + \cdots + a_s$ is divisible by n. (**Hint :** Consider $a_1,\ldots,a_n$ in the group $(\mathbb{Z}_n, +_n)$ and apply part (a).)

1.5 Let $n \in \mathbb{N}^*$. Show that:

(a) A residue class $[k]_n \in \mathbb{Z}_n$, $k \in \mathbb{Z}$, is invertible in the multiplicative monoid $(\mathbb{Z}_n, \cdot)$ if and only if $\gcd(k,n) = 1$, i. e. $(\mathbb{Z}_n, \cdot_n)^\times = \{[k]_n \mid \gcd(k,n) = 1\}$. In particular, the unit group $(\mathbb{Z}_n)^\times$ is a group of order $\varphi(n)$, where φ is the *Euler's totient* function, see Supplement S1.3. (**Hint :** Use the Bezout's Lemma Supplement S1.2 (d).) Compute the inverse of $[69]_{100}$ in $\mathbb{Z}_{100}$.

(b) $(\mathbb{Z}_n, +_n, \cdot_n)$ is a field if and only if n is a prime number.

[1] **Pigeonhole Principle** (Dirichlet): *For finite sets X, Y of the same cardinality and a map $f : X \to Y$, the following statements are equivalent* : (i) *f is injective.* (ii) *f is surjective.* (iii) *f is bijective.* .