E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Nikhil Gupta** (nikhil.gupta@csa.iisc.ernet.in ; **Lab No.: 303**) / **Vineet Nair** (vineetn90@gmail.com ; **Lab No.: 303**) / **Rahul Gupta** (rahul.gupta@csa.iisc.ernet.in ; **Lab No.: 224**) / **Sayantan Mukherjee** (meghanamande@gmail.com ; **Lab No.: 253**) / **Palash Dey** (palash@csa.iisc.ernet.in ; **Lab No.: 301 , 333 , 335**)

Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

3. Generating systems, Linear independence, Bases

Submit a solution of the $*$-Exercise ONLY. **Due Date :** Wednesday, 24-08-2011 (Before the Class)

Complete Correct solution of the Exercise 3.4 (b) carry BONUS POINTS!!!

3.1 **(a)** Let K be a field of characteristic $\neq 2$, i. e. $1+1\neq 0$ in K and let $a\in K$. Compute the solution set of the following systems of linear equations over K:

$$\begin{aligned} ax_1 + x_2 + x_3 &= 1 \\ x_1 + ax_2 + x_3 &= 1 \\ x_1 + x_2 + ax_3 &= 1\,; \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 + ax_3 &= 3 \\ x_1 + ax_2 + 3x_3 &= 2\,; \end{aligned}$$

For which a these systems have exactly one solution ?

(b) The set of m-tuples $(b_1,\ldots,b_m)\in K^m$ for which a linear system of equations $\sum_{j=1}^n a_{ij}x_j = b_i$, $i=1,\ldots,m$, over a field K has a solution is a K-subspace of K^m.

(c) Let K be a subfield of the field L and let $\sum_{j=1}^n a_{ij}x_j = b_i\,, i=1,\ldots,m$ be a system of linear equations over K. If this system has a solution $(x_1,\ldots,x_n)\in L^n$, then it also has a solution in K^n.

3.2 **(a)** Let $x_1,\ldots,x_n\in V$ be linearly independent (over K) in a K-vector space V and let $x := \sum_{i=1}^n a_ix_i \in V$ with $a_i\in K$. Show that $x_1-x,\ldots,x_n-x$ are linearly independent over K if and only if $a_1+\cdots+a_n\neq 1$.

(b) Let $x_1,\ldots,x_n$ be a basis of the K-vector space V and let $a_{ij}\in K$, $1\le i\le j\le n$. Show that

$$y_1 = a_{11}x_1\,,\ \ y_2 = a_{12}x_1 + a_{22}x_2\,,\ \ \ldots\ ,\ \ y_n = a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n$$

is a basis of V if and only if $a_{11}\cdots a_{nn}\neq 0$.

(c) Show that the family $\{\ln p \mid p \text{ prime number}\}$ of real numbers is linearly independent over $\mathbb{Q}$. (**Hint :** Use the Fundamental Theorem of Arithmetic, see Supplement S1.2 (f).)

3.3 Let K be a field and let $K[X]$ (respectively, $K[X]_m$, $m\in\mathbb{N}$) be the K-vector space of all polynomials (respectively, polynomials of degree $< m$) with coefficients in K. Let $f_n\in K[X]$, $n\in\mathbb{N}$, be a sequence of polynomials with $\deg f_n\le n$ for all $n\in\mathbb{N}$. Show that:

(a) For every $m\in\mathbb{N}$, $f_0,\ldots,f_{m-1}$ is a K-basis of the subspace $K[X]_m$ if and only if $\deg f_n = n$ for all $n=0,\ldots,m-1$. (**Hint :** Use Exercise 3.2 (b).)

(b) $f_n\,, n\in\mathbb{N}$, is a basis of the K-vector space $K[X]$ if and only if $\deg f_n = n$ for all $n\in\mathbb{N}$.

***3.4** **(a)** The geometric sequences $(1,\lambda,\lambda^2,\ldots,\lambda^n,\ldots)\in K^{\mathbb{N}}$, $\lambda\in K$, are linearly independent over K in the K-vector space of the sequences $K^{\mathbb{N}}$. (**Hint :** Let $y_j := (\lambda_j^{i-1})_{i\in\mathbb{N}^*}$. Suppose that $\lambda_1,\ldots,\lambda_n\in K$

are distinct and $\sum_{j=1}^{n} a_j y_j = 0$ with $a_1, \ldots, a_n \in K$. Then $a_1 x_1 + \cdots + a_n x_n = 0$, where $x_j := (1, \lambda_j, \ldots, \lambda_j^{n-1})$, and hence $a_1 = \cdots = a_n = 0$, See Supplement S3.4 (a).)

(b) Let $f : I \to K$ be a K-valued function with image $f(I)$ infinite. Then the family $f^n, n \in \mathbb{N}$ of powers of f is linearly independent (over K) in the K-vector space K^I of all K-valued function on the set I. (**Hint :** Since the image $f(I)$ of f is infinite, I is infinite. By restricting f to a suitable subset of I, we may assume that f is a sequence $f = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m, \ldots,)$ with pairwise distinct λ_m, $m \in \mathbb{N}$. Now, use the Supplement S3.4 (b).)

3.5 Let $F = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$ be an arbitrary polynomial of degree n and for $i \in \mathbb{N}$, let $F^{(i)}$ denote the i-derivative of F. Suppose that $m \cdot 1_K \neq 0$ for all $m = 1, \ldots, n$ (i. e., the characteristic $\operatorname{Char} K > n$ or 0). Then :

(a) The polynomials $F = F^{(0)}, F^{(1)}, \ldots, F^{(n)} = n! a_n$, is a basis of the K-vector space $K[X]_{n+1}$. (Recall that the (formal) k-th derivative (defined inductively)[1] $F^{(k)}(X) = \sum_{j=k}^{n} j(j-1)\cdots(j-k+1) a_j X^{j-k}$ of a polynomial $F = \sum_{i=0}^{n} a_i X^i \in K[X]$ is a polynomial of degree $n-k$ for every $k = 0, \ldots, n$ (if K is an arbitrary field of characteristic 0 or $> n$). — **Hint :** Use Exercise 3.2 (b).)

(b) If $\lambda_0, \ldots, \lambda_n$ are pairwise distinct, the the polynomials $F(X-\lambda_0), \ldots, F(X-\lambda_n) \in K[X]$ are linearly independent over K. (**Hint :** For this first we will prove the well-known Taylor's formula for polynomial functions:

$$F(X-\lambda) = \sum_{k=0}^{n} (-1)^k \lambda^k \frac{F^{(k)}(X)}{k!} .$$

By using *binomial formula*[2] and interchanging the summations, we get :

$$F(X-\lambda) = \sum_{j=0}^{n} a_j (X-\lambda)^j = \sum_{j=0}^{n} a_j \sum_{k=0}^{j} \binom{j}{k} X^{j-k} (-\lambda)^k = \sum_{k=0}^{n} (-1)^k \lambda^k \sum_{j=k}^{n} j(j-1)\cdots(j-k+1)\, a_j \frac{X^{j-k}}{k!}$$
$$= \sum_{k=0}^{n} (-1)^k \lambda^k \frac{F^{(k)}(X)}{k!} .$$

Now, to prove linear independence $F(X-\lambda_0), \ldots, F(X-\lambda_n)$ over K, consider $0 = \sum_{i=0}^{n} c_i F(X-\lambda_i)$ with coefficients $c_i \in K$. Using the above Taylor's formula, it follows that

$$0 = \sum_{i=0}^{n} c_i F(X-\lambda_i) = \sum_{i=0}^{n} c_i \sum_{k=0}^{n} (-1)^k \lambda_i^k \frac{F^{(k)}(X)}{k!} = \sum_{i=0}^{n} \frac{(-1)^k}{k!} \left(\sum_{i=0}^{n} c_i \lambda_i^k \right) F^{(k)}(X)$$

and hence by the linear independence of $F = F^{(0)}, F^{(1)}, \ldots, F^{(n)}$ over K (see part (a)) we have $\sum_{i=0}^{n} c_i \lambda_i^k = 0$ for all $k = 0, \ldots, n$. Now, use the Supplement S3.4 (a) to conclude that $c_0 = \cdots = c_n = 0$.)

[1] **Formal derivatives** Let K be a field. For a polynomial $F = \sum_{n \in \mathbb{N}} a_n X^n \in K[X]$, we define the (formal) derivative of F by $F' := \sum_{n \in \mathbb{N}} n a_n X^{n-1} \in K[X]$. This formal derivative satisfies usual product and quotient rules l $(FG)' = F'G + FG'$ for all $F, G \in K[X]$ and $(F/G)' = (GF' - G'F)/G^2$ for all $F, G \in K[X], G \neq 0$.

[2] **Binomial Formula :** For elements x and y in a commutative ring A and a natural number $n \in \mathbb{N}$, , we have $(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}$.