E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

Download from : http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Nikhil Gupta** (nikhil.gupta@csa.iisc.ernet.in ; **Lab No.: 303**) / **Vineet Nair** (vineetn90@gmail.com ; **Lab No.: 303**) / **Rahul Gupta** (rahul.gupta@csa.iisc.ernet.in ; **Lab No.: 224**) / **Sayantan Mukherjee** (meghanamande@gmail.com ; **Lab No.: 253**) / **Palash Dey** (palash@csa.iisc.ernet.in ; **Lab No.: 301 , 333 , 335**)

Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	Grade A	Grade B$^+$	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

6. Linear Maps and Bases ; — The Rank Theorem

Submit a solution of the $*$-Exercise ONLY. **Due Date :** Wednesday, 14-09-2016 (Before the Class)

Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

***6.1** Let V and W be finite dimensional K-vector spaces. Show that

(a) There is an injective K-homomorphism from V into W if and only if $\mathrm{Dim}_K V \leq \mathrm{Dim}_K W$. Deduce that a homogeneous linear system $\sum_{j=1}^{n} a_{ij} x_j = 0$, $i = 1, \ldots, m$ of m equations in n unknowns over K with $n > m$ has a non-trivial solution in K^n.

(b) There is a surjective K-homomorphism from V onto W if and only if $\mathrm{Dim}_K V \geq \mathrm{Dim}_K W$. Deduce that a linear system $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1, \ldots, m$ of m equations in n unknowns over K with $n < m$ has no solution in K^n for some $(b_1, \ldots, b_m) \in K^m$.

(c) A homogeneous linear system $\sum_{j=1}^{n} a_{ij} x_j = 0$, $i = 1, \ldots, n$ of n equations in n unknowns over K has a non-trivial solution in K^n if and only if at least one of the corresponding inhomogeneous system of linear equations $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1, \ldots, n$ has no solution in K^n.

6.2 Let f and g be endomorphisms of the finite dimensional K-vector space V. If $g \circ f$ is an automorphism of V, then show that both g and f are also automorphisms of V.

6.3 Let f be an operator on the finite dimensional K-vector space V. Show that the following statements are equivalent : (i) $\mathrm{Ker}\, f = \mathrm{Im}\, f$. (ii) $f^2 = 0$ and $\mathrm{Dim}_K V = 2 \cdot \mathrm{Rank}\, f$.

6.4 Let $f_i : V_i \to V_{i+1}$, $i = 1, \cdots, r$, be surjective K-vector space homomorphisms with finite dimensional kernels. Then the composition $f := f_r \circ \cdots \circ f_1$ from V_1 to V_{r+1} also has finite dimensional kernel and

$$\mathrm{Dim}_K \mathrm{Ker}\, f = \sum_{i=1}^{r} \mathrm{Dim}_K \mathrm{Ker}\, f_i\,.$$

(**Hint :** Proof by induction on r. For the inductive-step consider the K-linear map $\mathrm{Ker}\, f \to \mathrm{Ker}\, f_r \circ \cdots \circ f_2$ $x \mapsto f_1(x)$. Check that this map is surjective and apply the Rank-Theorem. — **Remark:** For example (see Supplement S3.18 and Supplement S5.5) : Let $P(X) = (X - \lambda_1) \cdots (X - \lambda_n)$ be a polynomial in $\mathbb{C}[X]$ with (not necessarily distinct) zeros $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then the differential operator $P(D) = (D - \lambda_1) \cdots (D - \lambda_n)$ on $\mathrm{C}^\infty_{\mathbb{C}}(I)$, where $I \subseteq \mathbb{R}$ is an interval has n-dimensional kernel, since for every $\lambda \in \mathbb{C}$, $D - \lambda$ is surjective (proof!) and has 1-dimensional kernel $\mathbb{C}e^{\lambda t}$. Moreover, if $\lambda_1, \ldots, \lambda_r$, $r \leq n$, are distinct zeros of $P(X)$ with multiplicities $n_1, \ldots, n_r$, then the quasi-polynomials $e^{\lambda_1 t}, te^{\lambda_1 t}, \ldots, t^{n_1 - 1} e^{\lambda_1 t}; \ldots ; e^{\lambda_r t}, te^{\lambda_r t}, \ldots, t^{n_r - 1} e^{\lambda_r t}$ are n linearly independent functions in $\mathrm{Ker}\, \mathbb{P}(D)$. In particular, they form a basis of $\mathrm{Ker}\, P(D)$ and is called a *fundamental system of solutions* of the differential equation $P(D)y = 0$.)