First Order Partial Differential Equation, Part - 2: Non-linear Equation

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First order non-linear equation

\[ F(x, y, u, p, q) = 0, \quad p = u_x, \quad q = u_y \]  

\[ F \in C^2(D_3), \text{ domain } D_3 \subset \mathbb{R}^5 \quad (1) \]

No directional derivative of \( u \) in \((x, y)\)-plane for a general \( F \).

Take a known solution \( u(x, y) \in C^2(D), \text{ } D \subset \mathbb{R}^2 \)

\[ F(x, y, u(x, y), p(x, y), q(x, y) = 0 \text{ in } D. \quad (2) \]
Charpit Equations

Taking \( x \) derivative

\[
F_x + F_u u_x + F_p p_x + F_q q_x = 0
\]

Using \( q_x = (u_y)_x = (u_x)_y = p_y \)

\[
F_p p_x + F_q p_y = -F_x - pF_u
\]  \( (3) \)

Beautiful, \( p \) is differentiated in the direction \((F_p, F_q)\).

Similarly

\[
F_p q_x + F_q q_y = -F_y - qF_u
\]  \( (4) \)
Charpit Equations contd..

Along one parameter family of curves in \((x, y)-\) plane given by

\[
\frac{dx}{d\sigma} = F_p, \quad \frac{dy}{d\sigma} = F_q
\]  

(5)

we have

\[
\frac{dp}{d\sigma} = -F_x - pF_u
\]  

(6)

\[
\frac{dq}{d\sigma} = -F_y - qF_u
\]  

(7)

Further

\[
\frac{du}{d\sigma} = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma} = pF_p + qF_q
\]  

(8)

Derived for a given solution \(u = u(x, y)\).
These, 5 equations for 5 quantities $x, y, u, p, q$ are complete irrespective of the solution $u(x, y)$. They are Charpit equations.

Given values $(u_0, p_0, q_0)$ at $(x_0, y_0)$, such that $(x_0, y_0, u_0, p_0, q_0) \in D_3$ ⇒ local unique solution of Charpit Equations with $(x_0, y_0, u_0, p_0, q_0)$ at $\sigma = 0$.

Autonomous system ⇒ 4 parameter family of solutions. $(x, y, u, p, q) = (x, y, u, p, q)(\sigma, c_1, c_2, c_3, c_4)$. 
Charpit Equations contd..

**Theorem.** The function $F$ is constant for every solution of the Charpit’s equations i.e. 
\[ F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = C(c_1, c_2, c_3, c_4) \] is independent of $\sigma$.

**Proof.** Simple,
\[
\frac{dF}{d\sigma} = \frac{dx}{d\sigma} F_x + \frac{dy}{d\sigma} F_y + \frac{du}{d\sigma} F_u + \frac{dp}{d\sigma} F_p + \frac{dq}{d\sigma} F_q = 0 \quad (9)
\]
when we use Charpit’s equations. In order that solution of Charpit’s equations satisfies the relation $F = 0$, choose $c_1, c_2, c_3$ and $c_4$ such that
\[
C(c_1, c_2, c_3, c_4) = 0 \implies c_4 = c_4(c_1, c_2, c_3) \quad (10)
\]
Monge strip and characteristic curves

**Monge strip** is a solution of the Charpit’s equations satisfying

$$F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma) = 0.$$  \hspace{1cm} (11)

It is a 3 parameter family of functions

$$(x, y, u, p, q)(\sigma, c_1, c_2, c_3) = 0$$  \hspace{1cm} (12)

**Characteristic curves:** From the Monge strips, the curves $x(\sigma, c_1, c_2, c_3), y(\sigma, c_1, c_2, c_3)$ in $(x, y)$-plane are 3-parameter family of characteristic curves.
Cauchy data

- $u(x, y) : D \rightarrow \mathbb{R}$
- $\gamma : (x = x_0(\eta), y = y_0(\eta))$ is curve in $D$.
- $u_0(\eta) = u(x_0(\eta), y_0(\eta))$

![Diagram of Cauchy problem solution with characteristic curves $C_c$.](image)

**Fig. 1.1. Solution of a Cauchy problem with the help of characteristic curves $C_c$.**
Value of $u$ is carried along a characteristic not independently but together with values of $p$ and $q$.

We need values of $x_0, y_0, u_0, p_0$ and $q_0$ at $P_0$ on datum curve as initial data for Charpit’s equations.
Solution of a Cauchy problem contd..

How to get values of $p_0$ and $q_0$ from Cauchy data?

$$u = u_0(\eta) \text{ on } \gamma : x = x_0(\eta), y = y_0(\eta)$$  \hspace{1cm} (13)

First we have

$$F(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)) = 0.$$  \hspace{1cm} (14)

Differentiating $u_0(\eta) = u(x_0(\eta), y_0(\eta))$ wrt $\eta$, we get

$$u'(\eta) = p_0(\eta)x'_0(\eta) + q_0y'_0(\eta).$$  \hspace{1cm} (15)

Solve now $p_0(\eta)$ and $q_0(\eta)$. Cauchy data for Charpit’s ODEs

$$(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) \mid_{\sigma=0} = (x_0, y_0, u_0, p_0, q_0)(\eta)$$  \hspace{1cm} (16)
Solution of a Cauchy problem contd..

Solve the Charpit’s equations

\[
\frac{dx}{d\sigma} = F_p, \quad \frac{dy}{d\sigma} = F_q, \quad \frac{du}{d\sigma} = pF_p + qF_q \quad (17)
\]

\[
\frac{dp}{d\sigma} = -(F_x + pF_u), \quad \frac{dq}{d\sigma} = -(F_y + qF_u) \quad (18)
\]

with above initial conditions:

\[
x = x(\sigma, \eta), \quad y = y(\sigma, \eta) \quad (19)
\]

\[
u = u(\sigma, \eta), \quad p = p(\sigma, \eta), \quad q = q(\sigma, \eta) \quad (20)
\]

Then solve \( \sigma = \sigma(x, y), \eta = \eta(x, y) \) from (20) and get \( u(x, y) = u(\sigma(x, y), \eta(x, y)) \).

We can also get \( u_x = p(x, y), u_y = q(x, y) \)
Solution of a Cauchy problem contd..

Theorem

1. $F(x, y, u, p, q) \in C^2(D_3)$, domain $D_3 \subset \mathbb{R}^5$
2. $x_0(\eta), y_0(\eta), u_0(\eta) \in C^2(I)$, $\eta \in I \subset \mathbb{R}$
3. $(x_0(\eta), y_0(\eta), u_0(\eta)p_0(\eta), q_0(\eta)) \in D_3$, $\eta \in I$
4. $p_0(\eta), q_0(\eta) \in C^1(I)$, $\eta \in I$
5. $\frac{d x_0}{d \eta} F_q(x_0, y_0, u_0, p_0, q_0) - \frac{d y_0}{d \eta} F_p(x_0, y_0, u_0, p_0, q_0) \neq 0$

$\Rightarrow$ There exist a unique solution of the Cauchy problem such that

$$u(x, y)|_{\gamma} = u_0, \quad p(x, y)|_{\gamma} = p_0, \quad q(x, y)|_{\gamma} = q_0 \quad (21)$$
Important point for the existence and uniqueness of the Cauchy problem is that the datum curve $\gamma$ is nowhere tangential to a characteristic curve.

If $\gamma$ is a characteristic curve, the data $u_0(\eta)$ is to be restricted (i.e., the equations of $p_0$ and $q_0$ are also satisfied) and when this restriction is imposed, the solution of the Cauchy problem is non-unique - infinity of solutions exist.
Isotropic wave motion with constant velocity

Consider an isotropic wave moving into a uniform medium with constant velocity \( c \) (like light wave)
Isotropic wave motion with constant velocity contd..

Let a wavefront in such a wave be \( u(x, y) = ct \).

\[ c(t + \delta t) = u(x + \delta x, y + \delta y) \]

Taylor expansion of \( u \) up to first order terms (using \( ct = u(x, y) \))

\[
\frac{c}{\sqrt{u_x^2 + u_y^2}} \delta t = \frac{u_x}{\sqrt{u_x^2 + u_y^2}} \delta x + \frac{u_y}{\sqrt{u_x^2 + u_y^2}} \delta y = n_1 \delta x + n_2 \delta y = \delta n
\]

(22)

\[
\frac{c}{\sqrt{u_x^2 + u_y^2}} = \frac{\delta n = \text{normal displacement}}{\delta t} = c
\]

(23)

\[ \Rightarrow p^2 + q^2 = 1; \quad p = u_x, q = u_y \]

(24)
Isotropic wave motion with constant velocity contd..

**Problem.** Find successive positions of the wavefront \( u(x, y) = ct \) when the initial position is

\[
\alpha x + \beta y = 0, \quad \alpha^2 + \beta^2 = 1, \quad \text{where } u = 0
\]  

(25)

**Cauchy problem**

\[
F \equiv p^2 + q^2 = 1
\]

\[
\gamma : x_0 = \beta \eta, \quad y_0 = -\alpha \eta
\]

cauchy data

\[
u_0 = 0
\]

**Values of** \( p_0 \) **and** \( q_0 \)

\[
p_0^2 + q_0^2 = 1, \quad \beta p_0 - \alpha q_0 = 0
\]

\[
\Rightarrow p_0 = \pm \alpha, \quad q_0 = \pm \beta
\]  

(26)
Isotropic wave motion with constant velocity contd..

Solution of Charpit’s equations

\[
\begin{align*}
\frac{dx}{d\sigma} &= 2p, \quad \frac{dy}{d\sigma} = 2q, \\
\frac{du}{d\sigma} &= 2(p^2 + q^2) = 2, \quad \frac{dp}{d\sigma} = 0, \quad \frac{dq}{d\sigma} = 0
\end{align*}
\]  
(27)

Solution are

\[
\begin{align*}
x &= \pm 2\alpha \sigma + \beta \eta, \quad y = \pm 2\beta \sigma - \alpha \eta \\
u &= 2\sigma, \quad p = \pm \alpha, \quad q = \pm \beta
\end{align*}
\]
⇒ \[
\sigma = \pm \frac{1}{2}(\alpha x + \beta y)
\]
⇒ \[
u = \pm (\alpha x + \beta y)
\]
(28)

Two solutions of 2 problems, but uniqueness theorem not violated.
Isotropic wave motion with constant velocity contd..

Wavefronts

\[ \alpha x + \beta y = \pm ct \]  \hspace{1cm} (29)

+ sign for forward propagating wavefront

- sign for backward propagating wavefront

Normal distance at time \( t \) from the initial position = \( \pm ct \)
We have presented the theory of characteristics of first order PDEs briefly. It is based on the existence of characteristics curves in the \((x, y)\)-plane.

Along each of these characteristics we derive a number of compatibility conditions, which are transport equations and which are sufficient to carry all necessary information from the datum curve in the Cauchy problem into a domain in which solution is determined.

In this sense every first order PDE is a hyperbolic equation.
We have omitted a special class of solutions known as complete integral, for which any standard text may be consulted.

Every solution of the PDE (1) can be obtained from a complete integral.

We can also solve a Cauchy problem with its help.

Complete integral plays an important role in Physics.
So far we have discussed only a genuine solution, which is valid locally.

- We have seen that characteristic carry information about the solution. Characteristic curves are the only curve which can sustain discontinuities of certain types in the solution. For a linear equation the discontinuities can be in the solution and its derivatives, for a quasilinear equation the discontinuities can be in the first and higher order derivatives and for nonlinear equations the discontinuities can be in second and higher order derivatives.
It is possible to develop a theory of first order PDE starting from the definition of characteristic curves as the curves which carry above type of discontinuities. See P. Prasad, A theory of first order PDE through propagation of discontinuities, RMS Mathematics News letter, 2000, 10, 89-103.

Absence of real characteristic curves of the equation $u_x + iu_y = 0$ in $(x, y)$-plane shows that its solution can not have discontinuities of any type along a curve in $(x, y)$-plane. This is related to the regularity of a solution of an elliptic PDE.
There is a fairly complete theory of weak solutions of Hamilton-Jacobi equations, a particular case of the nonlinear equation (1). Generally the domain of validity of a weak solution with Cauchy data on the $x$-axis is at least half of the $(x, y)$-plane.

Theory of a single conservation law, a first order equation, is particularly interesting not only from the point of view of theory but also from the point of view of applications (Prasad, 2001).
1 Consider the partial differential equation

\[ F \equiv u(p^2 + q^2) - 1 = 0. \]

(i) Show that the general solution of the Charpit’s equations is a four parameter family of strips represented by

\[
\begin{align*}
x &= x_0 + \frac{2}{3} u_0 (2\sigma)^{\frac{3}{2}} \cos \theta, \\
y &= y_0 + \frac{2}{3} u_0 (2\sigma)^{\frac{3}{2}} \sin \theta, \\
u &= 2u_0 \sigma, \\
p &= \frac{\cos \theta}{\sqrt{2\sigma}}, \\
q &= \frac{\sin \theta}{\sqrt{2\sigma}}
\end{align*}
\]

where \( x_0, y_0, u_0 \) and \( \theta \) are the parameters.

(ii) Find the three parameter sub-family representing the totality of all Monge strips.

(iii) Show that the characteristic curves consist of all straight lines in the \((x, y)\)-plane.
2 Solve the following Cauchy problems:

(i) \( \frac{1}{2}(p^2 + q^2) = u \) with Cauchy data prescribed on the circle \( x^2 + y^2 = 1 \) by

\[
u(\cos \theta, \sin \theta) = 1, \quad 0 \leq \theta \leq 2\pi\]

(ii) \( p^2 + q^2 + (p - \frac{1}{2}x) (q - \frac{1}{2}y) - u = 0 \) with Cauchy data prescribed on the \( x \)-axis by

\[
u(x, 0) = 0\]

(iii) \( 2pq - u = 0 \) with Cauchy data prescribed on the \( y \)-axis by

\[
u(0, y) = \frac{1}{2}y^2\]

(iv) \( 2p^2x + qy - u = 0 \) with Cauchy data on \( x \)-axis

\[
u(x, 1) = -\frac{1}{2}x.\]


2. P. Prasad. A theory of first order PDE through propagation of discontinuities. Ramanujan Mathematical Society News Letter, 2000, **10**, 89-103; see also the webpage:

Thank You!