KINEMATICAL CONSERVATION LAWS: EQUATIONS OF EVOLUTION OF CURVES AND SURFACES

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Outline

1. Introduction
   - Evolution of Surfaces in $\mathbb{R}^m$
   - Different Methods Used in Computation

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1. Introduction
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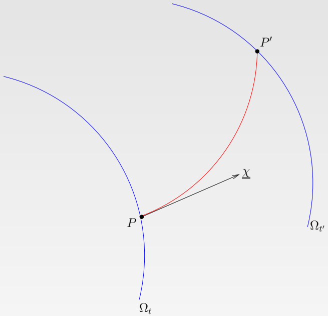
Evolution of a Surface

Consider a moving surface Ω_t in $\mathbb{R}^m$

$$\Omega_t\colon \varphi(\mathbf{x}, t) = 0. \qquad (1)$$



Let there be a law, which determines the motion of each point of Ω_t and for which

$$P_{|\Omega_t} \longrightarrow P'_{|\Omega_{t'}}$$

contd..

- The path of $P(\mathbf{x}(t))$ is a **ray** and with $\mathbf{n}$ = unit normal to Ω_t $=\frac{\nabla\varphi}{|\nabla\varphi|}$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \chi(\mathbf{x}, t; \mathbf{n}), \tag{2}$$

is the ray velocity for the given law.

- $\varphi(\mathbf{x}, t)$ satisfies the eikonal equation

$$\varphi_t + \langle \chi, \nabla\varphi \rangle = 0, \tag{3}$$

or

$$\varphi_t + m|\nabla\varphi| = 0, \tag{4}$$

where $m(\mathbf{x}, t, \mathbf{n}) = \frac{-\varphi_t}{|\nabla\varphi|}$ = normal velocity of $\Omega_t = \langle \mathbf{n}, \chi \rangle$

Available methods

(A) Level Set Method (LSM):
solving directly the equation (3).

(B) Ray theory:
The Charpit's equations $\Rightarrow$ equations for $\mathbf{x}$ and φ_{x_α}
$\Rightarrow$ Hamilton canonical equations.
$\mathbf{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$ can be used to derive ray equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \chi \tag{5}$$

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = -n_\beta n_\gamma \left(\frac{\partial}{\partial \eta^\alpha_\beta} \chi_\gamma \right), \tag{6}$$

where $\frac{\partial}{\partial \eta^\alpha_\beta} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta}$ are tangential derivatives on Ω_t.

contd..

provided χ satisfies

$$n_\beta n_\gamma \left(n_\beta \frac{\partial}{\partial \eta_\alpha} - n_\alpha \frac{\partial}{\partial \eta_\beta} \right) \chi_\gamma = 0, \tag{7}$$

for every $\alpha = 1, 2, \ldots, m$ and $\frac{\partial}{\partial \eta^\alpha_\beta}$ does not operate on $\mathbf{n}$.

Note

Summation convention on α, β, γ *on range* $1, 2, \ldots, m$.

Note

Both LSM and ray theory involve equations in $m+1$ *independent variables* $(x_1, ..., x_m, t)$.

contd..

(C) Kinematical Conservation Laws (KCL):

(i) This involves solving equations in m independent variables - ray coordinates $(\xi, t) = (\xi_1, \xi_2, \ldots, \xi_{m-1}, t)$.

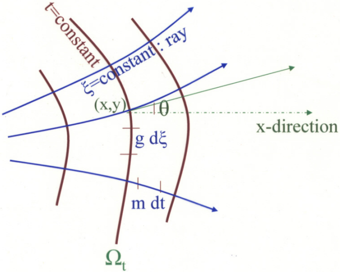
(ii) A general theory is available, but assume the direction of $\mathbf{n}$ and χ are same.

(iii) To start with $m = 2$, i.e, $(x, y)-$plane.

(iv) $\Rightarrow$ 2-D KCL in (ξ, t).

A Ray Coordinate System for 2-D KCL

Ray coordinate system associated with $\Omega_t : (\xi, t)$



(x, y) is a point on the moving curve Ω_t at time t.

θ= angle which the ray at (x, y) makes with x-direction.

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Notations

We have assumed that rays associated with Ω_t are orthogonal to Ω_t.

$$\begin{aligned} m &= \text{suitably non-dimensionalised velocity of propagation of } \Omega_t, \\ &= \text{Mach number of } \Omega_t. \\ \xi &= \text{constant are rays.} \\ t &= \text{constant are successive positions of } \Omega_t. \\ g\mathrm{d}\xi &= \text{element of length along wavefront} : \Omega_t. \\ m\mathrm{d}t &= \text{element of length along a ray.} \end{aligned}$$

Locally, $(x, y) \Leftrightarrow (\xi, t)$.

2-D KCL

$$\frac{1}{g}\frac{\partial}{\partial \xi} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} \quad (8)$$

The displacement $(\mathrm{d}x, \mathrm{d}y)$ of a point P in the (x, y)-space due to an arbitrary displacement $(\mathrm{d}\xi, \mathrm{d}t)$ in (ξ, t)-space is given by

$$\mathrm{d}\mathbf{x} = g\mathbf{u}\mathrm{d}\xi + m\mathbf{n}\mathrm{d}t \quad (9)$$

or

$$\mathrm{d}x = -g\sin\theta\mathrm{d}\xi + m\cos\theta\mathrm{d}t, \quad (10)$$

$$\mathrm{d}y = g\cos\theta\mathrm{d}\xi + m\sin\theta\mathrm{d}t. \quad (11)$$

$x_{t\xi} = x_{\xi t}$, $y_{t\xi} = y_{\xi t} \Rightarrow$ 2-D KCL (Morton, Prasad, Ravindran, 1992)

$$(g\sin\theta)_t + (m\cos\theta)_\xi = 0, \quad (12)$$

$$(g\cos\theta)_t - (m\sin\theta)_\xi = 0. \quad (13)$$

R-H Conditions

KCL is a system of conservation laws. Its solution can have shocks in (ξ, t) plane

$$\text{shock velocity:} \quad K = \frac{\mathrm{d}\xi_s(t)}{\mathrm{d}t} \tag{14}$$

and jump relations

$$-Kg_- \sin\theta_- + m_- \cos\theta_- = -Kg_+ \sin\theta_+ + m_+ \cos\theta_+, \tag{15}$$

$$Kg_- \cos\theta_- + m_- \sin\theta_- = Kg_+ \cos\theta_+ + m_+ \sin\theta_+. \tag{16}$$

$\Rightarrow$ Hugoniot relation

$$\cos(\theta_- - \theta_+) = \frac{m_- g_- + m_+ g_+}{m_- g_+ + m_+ g_-} \tag{17}$$

and shock velocity

$$K = \pm \left(\frac{m_-^2 - m_+^2}{g_+^2 - g_-^2} \right)^{1/2}. \tag{18}$$

Geometry of rays and Ω_t on two sides of a kink line

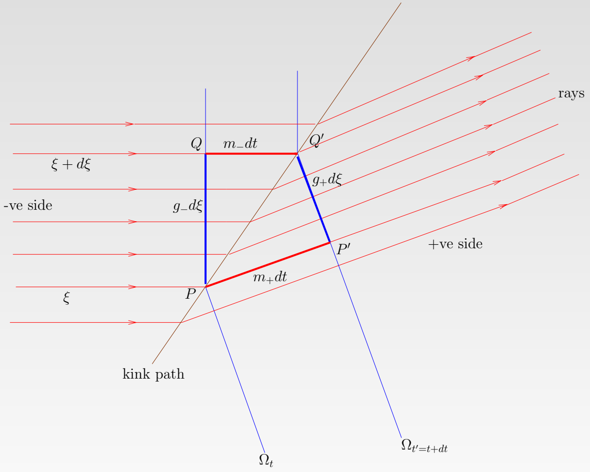


Figure: Kink phenomenon

Conservation of Distance

Using Pythagorous theorem

$$m_+(\mathrm{d}t)^2 + g_+(\mathrm{d}\xi)^2 = (PQ')^2 = m_-^2(\mathrm{d}t)^2 + g_-(\mathrm{d}\xi)^2 \tag{19}$$

$\Rightarrow$ same expression for $K = \pm\left(\frac{m_-^2 - m_+^2}{g_+^2 - g_-^2}\right)^{1/2}$.

We can prove even a theorem on conservation of distance (we state for 3-D KCL)

Theorem

The jump relations imply conservation of distance in x_1, x_2 and x_3 directions (and hence in any arbitrary direction in $\mathbf{x}$-space) in the sense that the expressions for a vector displacement $(d\mathbf{x})_{\mathscr{K}_t}$ of a point of the kink line $\mathscr{K}_t$ in an infinitesimal time interval dt, when computed in terms of variables on the two sides of a kink surface, have the same value. This displacement of the point is assumed to take place on the kink surface and that of its image in (ξ_1, ξ_2, t)-space takes place on the shock surface such that the corresponding displacement in (ξ_1, ξ_2)-plane is with the shock front (i.e., it is in direction $\frac{d}{dt}(\xi_1, \xi_2) = (E_1, E_2)K$).

KCL - undetermined system

- KCL is an under-determined system: two equations for three variables m, θ and g.
- This is a consequence of our assumption of the existence of χ without stating the nature of the dynamics of Ω_t.
- We mention three ways to close the KCL
- $$m = \text{a known function of } \mathbf{n} \text{ (or curvature } \kappa).$$
- $$\chi = m\mathbf{n},\ m \text{ independent of } \mathbf{n}.$$
  $$\left(gG^{-1}(m)\right)_t = 0, \tag{20}$$
  Baskar and Prasad have studied Riemann problem for 2-D KCL with a general $G(m)$.
- For a shock front, there is an infinite system of closure equations.

KCL - 3 closures explained

- When Ω_t is a surface of crystal growing in a medium, one model is

$$m = \text{a known function of } \mathbf{n} \text{ (or curvature } \kappa).$$

- When Ω_t is any surfce of transition or separation, as in oil recovery process, one needs to find appropriate closure relation and the KCL method will be applicable.
- One of the most important applications has been the sonic boom. KCL method is the only method which gives a neat mathematical formulation, where theorems can be proved. $\Rightarrow$ Ill posed Cauchy problem.
- For wave propagation in a polytropic gas satisfying Fermat's principle of stationary time of transit, motion of Ω_t is isotropic, i.e,

$$\chi = m\mathbf{n}, \; m \text{ independent of } \mathbf{n}.$$

Remarks contd..

and we may close KCL by a conservation law

$$\left(gG^{-1}(m)\right)_t = 0, \tag{21}$$

where

$$G(m) = (m-1)^{-2}e^{-2(m-1)}. \tag{22}$$

- Similarly for the motion of a crest line of a curved solitary wave on the surface of shallow water, we can deduce

$$G(m) = (m-1)^{-\frac{3}{2}}e^{-\frac{3}{2}(m-1)}. \tag{23}$$

Mapping onto the physical space

After solving the conservation equations of KCL in (ξ, t)-plane, the geometry of the wavefront Ω_t can be obtained by solving the ray equations

$$x_t = m\cos\theta, \quad y_t = m\sin\theta \tag{24}$$

or

$$x_\xi = -g\sin\theta, \quad y_\xi = g\cos\theta. \tag{25}$$

Difference in solutions: Linear Ray Theory and WNLRT with KCL

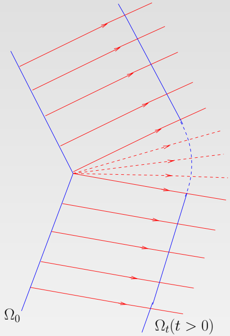


Figure: Linear wavefront produced by a convex wedge shaped piston

.... wavefront from corner Huygen's method.
— wavefront from the smooth part by ray theory or Huygen's method.
— rays generated from the corner fill the empty wedged shape region.

Nonlinear wavefront - exact solution in (ξ, t)-plane

WNLRT with KCL

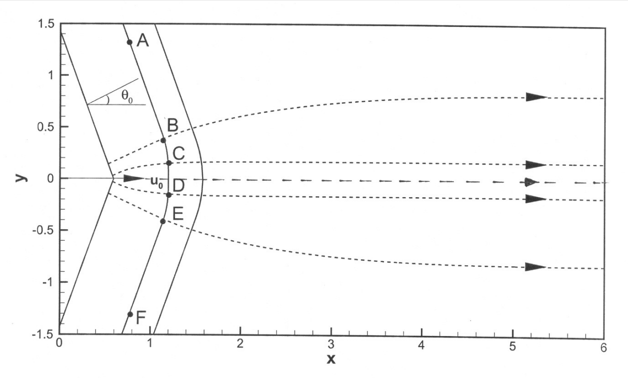


Figure: Nonlinear wavefront produced by a piston, $m = 1.05$

Linear wavefront

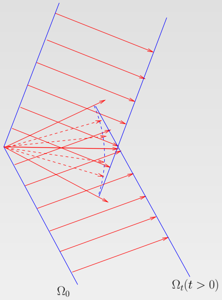
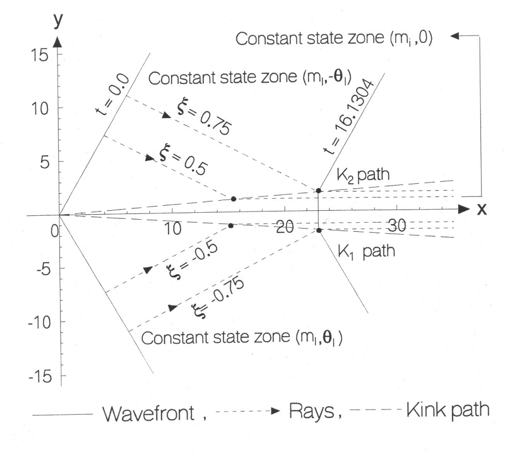


Figure: Linear wavefront produced by a concave wedge shaped piston.

..... wavefront produced by the corner O by Huygen's method. This is an arc of a circle with centre at O.
—— wavefront produced by the smooth part of the initial wavefront by ray theory or Huygen's method.

Resolution of caustic

Exact solution of 2-D KCL: Resolution of the linear caustic by non-linearity



Resolution of caustic contd.

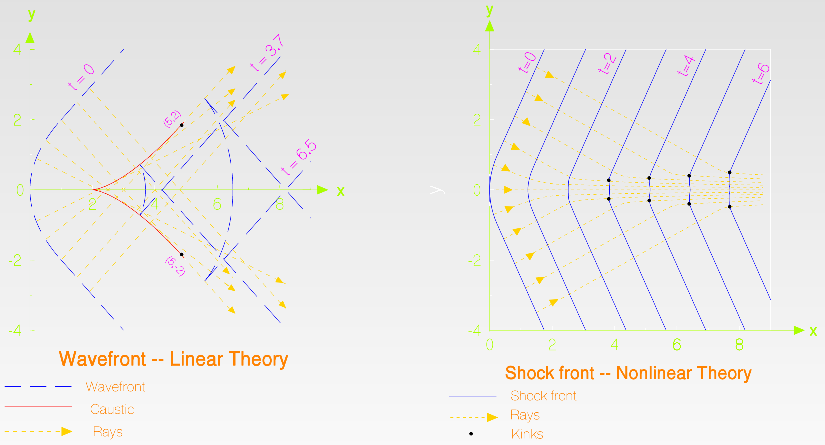
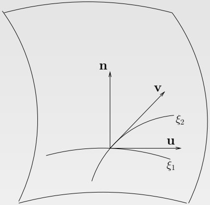


Figure: Resolution of the caustic by nonlinearity

3-D KCL

KCL for a surface Ω_t in $\mathbb{R}^3$



Two families of curves $\xi_2 =$ constant and $\xi_1 =$ constant on Ω_t.
$\mathbf{u} =$ unit tangent vector in the direction ξ_1 increasing and $\xi_2 =$ const.
$\mathbf{v} =$ unit tangent vector in the direction ξ_2 increasing and $\xi_1 =$ const.
$\mathbf{n} =$ unit normal to $\Omega_t = \frac{\nabla\varphi}{|\nabla\varphi|}$.
$m = \frac{-\varphi_t}{|\nabla\varphi|}$, normal velocity of Ω_t.

Notations

- $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is a right handed system

$$\mathbf{n} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}, \ \langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle = 1.$$

- g_1 and g_2 are metrics associated with variables ξ_1 and ξ_2.
- m = velocity of Ω_t in $\mathbf{n}$ direction.
- Isotropic case:

$$\text{Ray velocity, } \chi = m\mathbf{n},$$

$$\text{Rays, } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \chi = m\mathbf{n}.$$

3-D KCL derivation

Displacement of a point P in $\mathbf{x}$-space due to an arbitrary displacement in (ξ_1, ξ_2, t)-space is

$$\mathrm{d}\mathbf{x} = g_1\mathbf{u}\mathrm{d}\xi_1 + g_2\mathbf{v}\mathrm{d}\xi_2 + m\mathbf{n}\mathrm{d}t \tag{26}$$

Existence of the two families of curves and rays give the KCL

$$(g_1\mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0, \tag{27}$$

$$(g_2\mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0 \tag{28}$$

and

$$(g_2\mathbf{v})_{\xi_1} - (g_1\mathbf{u})_{\xi_2} = 0. \tag{29}$$

Theorem

If the third system is satisfied at $t = 0$, then the first two systems imply that the third system is satisfied for $t > 0$.

Derivation contd..

Finally the 3-D KCL is obtained as (Giles, Prasad and Ravindran, 1996)

$$(g_1\mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0, \tag{30}$$

$$(g_2\mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0, \tag{31}$$

with ξ_1 and ξ_2 so chosen that

$$(g_2\mathbf{v})_{\xi_1} - (g_1\mathbf{u})_{\xi_2} = 0 \text{ at } t = 0. \tag{32}$$

The mapping from (ξ_1, ξ_2, t)-space to (x_1, x_2, x_3)-space is given by the ray equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = m\mathbf{n}. \tag{33}$$

The number of dependent variables in 6 KCL equations	
$\mathbf{u} = (u_1, u_2, u_3)$	only 2
$\mathbf{v} = (v_1, v_2, v_3)$	only 2
(g_1, g_2)	2
m	1
Total	7

Differential form of KCL

Conservation laws are necessary for any computation but their differential forms are required to study many properties. For this we choose dependent variables as $(u_1, u_2, v_1, v_2, m, g_1, g_2)$. Lengthy and difficult calculations lead to equations

$$g_1 u_{1t} - n_1 m_{\xi_1} + b_{11}^{(1)} u_{1\xi_1} + b_{12}^{(1)} u_{2\xi_1} + b_{13}^{(1)} v_{1\xi_1} + b_{14}^{(1)} v_{2\xi_1} = 0, \quad (34)$$

$$g_1 u_{2t} - n_2 m_{\xi_1} + b_{21}^{(1)} u_{1\xi_1} + b_{22}^{(1)} u_{2\xi_1} + b_{23}^{(1)} v_{1\xi_1} + b_{24}^{(1)} v_{2\xi_1} = 0, \quad (35)$$

$$g_2 v_{1t} - n_1 m_{\xi_2} + b_{31}^{(2)} u_{1\xi_2} + b_{32}^{(2)} u_{2\xi_2} + b_{33}^{(2)} v_{1\xi_2} + b_{34}^{(2)} v_{2\xi_2} = 0, \quad (36)$$

$$g_2 v_{2t} - n_2 m_{\xi_2} + b_{41}^{(2)} u_{1\xi_2} + b_{42}^{(2)} u_{2\xi_2} + b_{43}^{(2)} v_{1\xi_2} + b_{44}^{(2)} v_{2\xi_2} = 0, \quad (37)$$

$$g_{1t} + b_{61}^{(1)} u_{1\xi_1} + b_{62}^{(2)} u_{2\xi_1} = 0, \quad (38)$$

$$g_{2t} + b_{73}^{(2)} v_{1\xi_2} + b_{74}^{(2)} v_{2\xi_2} = 0. \quad (39)$$

The expressions for the coefficients are very lengthy and simply writing here makes no sense! but plays some role later!!

Equivalence of KCL and ray equations for smooth solutions

Theorem

For a given smooth function m of $\mathbf{x}$ and t, the ray equations

$$\frac{d\mathbf{x}}{dt} = m\mathbf{n}, \quad |\mathbf{n}| = 1, \tag{40}$$

$$\frac{d\mathbf{n}}{dt} = -\mathbf{L}m := -\{\nabla - \mathbf{n}(\mathbf{n}, \nabla)\}m \tag{41}$$

are equivalent to the KCL

$$(g_1\mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0, \tag{42}$$

$$(g_2\mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0, \tag{43}$$

with

$$(g_2\mathbf{v})_{\xi_1} - (g_1\mathbf{u})_{\xi_2} = 0 \quad at \quad t = 0. \tag{44}$$

Proof of the theorem

Ray equations $\Rightarrow$ KCL

- Ray equations $\Rightarrow$ successive positions of Ω_t.
- On Ω_t, choose ξ_1 and ξ_2 with metrics g_1, g_2 and $\mathbf{u}$ and $\mathbf{v}$ respectively.
- Conditions for the existence of ξ_1 and $\xi_2 \Rightarrow$ KCL.

Proof is complete.

Note

Direct derivation of the differential equations from

$$g_1 = \sqrt{{x_1}_{\xi_1}^2 + {x_2}_{\xi_1}^2 + {x_3}_{\xi_1}^2} \quad \textit{and} \quad g_2 = \sqrt{{x_1}_{\xi_2}^2 + {x_2}_{\xi_2}^2 + {x_3}_{\xi_2}^2} \tag{45}$$

is also simple.

Proof contd..

KCL $\Rightarrow$ Ray equations

- Smooth vector field $\mathbf{u}, \mathbf{v}, \mathbf{n}$ and smooth scalar functions m, g_1, g_2 in (ξ_1, ξ_2, t)-space, satisfying

$$\langle \mathbf{n}, \mathbf{u} \rangle = 0, \ \langle \mathbf{n}, \mathbf{v} \rangle = 0 \tag{46}$$

  and KCL are given.
- By the fundamental integrability theorem (Courant & John), KCL $\Rightarrow$ existence of a vector $\mathbf{x}$ such that

$$(\mathbf{x}_t, \mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) = (m\mathbf{n}, g_1\mathbf{u}, g_2\mathbf{v}). \tag{47}$$

- Due to (46) Jacobian $\neq 0$, hence (47) implies $\mathbf{x}$-space $\Leftrightarrow$ (ξ_1, ξ_2, t)-space.

Proof contd..

- $t =$ constant curve in (ξ_1, ξ_2, t)-space $\Rightarrow \Omega_t$ with ξ_1 and ξ_2 as coordinates and $\mathbf{u}, \mathbf{v}$ as tangent vectors $\Rightarrow \mathbf{n}$ is orthogonal to Ω_t,. Then first of (47) is the first ray equation.
- Let Ω_t be represented by $\varphi(\mathbf{x}, t) = 0$, then the first ray equation $\Rightarrow \varphi$ satisfies the pde

$$\varphi_t + m|\nabla\varphi| = 0$$

$\Rightarrow$ The second part of ray equations.
Proof is complete.

Note

Proof of direct derivation of the equation for $\mathbf{n}$ *form the differential from of KCL involves very long calculations!.*
Here the long expressions for the coefficients makes sense!!

WNLRT in a polytropic gas

Both systems: Ray equations and KCL are not complete, i.e., they are under-determined as there is no equation for m.

One way of closing: **Energy transport equation from WNLRT**

$$\left(g_1 g_2 (m-1)^2 e^{2(m-1)} \sin\chi\right)_t = 0, \tag{48}$$

where $\langle \mathbf{u}, \mathbf{v} \rangle = \cos\chi$.

KCL + Energy equation = conservation form of WNLRT.

Matrix form of the differential equations of KCL

Let $\mathbf{U} = (u_1, u_2, v_1, v_2, m, g_1, g_2)^T$. The differential form of KCL and energy transport equation for $\mathbf{U}$ are

$$A\mathbf{U}_t + B^{(1)}\mathbf{U}_{\xi_1} + B^{(2)}\mathbf{U}_{\xi_2} = 0, \tag{49}$$

where $A, B^{(1)}$ and $B^2 \in \mathbb{R}^{7\times 7}$.

Why these?
To study eigenvalues and eigenvectors.

Special form of the energy equation and second part of the ray equations

The second ray equation is

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = -\mathbf{L}m. \tag{50}$$

Define

$$\frac{\partial}{\partial \lambda_1} = n_1\frac{\partial}{\partial x_3} - n_3\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial \lambda_2} = n_3\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_3}. \tag{51}$$

$$\begin{aligned}\frac{\partial n_1}{\partial t} - \frac{n_2^2+n_3^2}{n_3}\frac{\partial m}{\partial \lambda_1} - \frac{n_1 n_2}{n_3}\frac{\partial m}{\partial \lambda_2} &= 0,\\ \frac{\partial n_2}{\partial t} + \frac{n_1 n_2}{n_3}\frac{\partial m}{\partial \lambda_1} + \frac{n_1^2+n_3^2}{n_3}\frac{\partial m}{\partial \lambda_2} &= 0,\\ \frac{\partial m}{\partial t} - \frac{(m-1)}{2n_3}\frac{\partial n_1}{\partial \lambda_1} + \frac{(m-1)}{2n_3}\frac{\partial n_2}{\partial \lambda_2} &= 0,\end{aligned} \tag{52}$$

Where λ_1 and λ_2 are not variables but $\frac{\partial}{\partial \lambda_1}$ and $\frac{\partial}{\partial \lambda_2}$ are symbols for operators.

A useful result

Result

Let $P_0(\mathbf{x}_0)$ be a given point on Ω_t. Then there exist two one parameter families of smooth curves on Ω_t such that the unit vectors $\mathbf{u}_0$ and $\mathbf{v}_0$ along the members of the curves at the chosen point P_0 can have any two arbitrary directions and the metrics g_{10} and g_{20} at this point can have any two positive values.

A frozen form of equations

At the point $P_0(\mathbf{x}_0)$ at time t, we choose

$$\mathbf{u}_0 = \frac{1}{\sqrt{n_1^2 + n_3^2}}(-n_3, 0, n_1), \quad \mathbf{v}_0 = \frac{1}{\sqrt{n_3^2 + n_2^2}}(0, n_3, -n_2). \tag{53}$$

and $g_1 = g_{10}$ and $g_2 = g_{20}$, arbitrary values, then

$$\frac{1}{g_{10}}\frac{\partial}{\partial \xi_1} = \langle \mathbf{u}_0, \nabla \rangle, \quad \frac{1}{g_{20}}\frac{\partial}{\partial \xi_2} = \langle \mathbf{v}_0, \nabla \rangle. \tag{54}$$

Frozen form contd..

The system (52) leads to frozen equations at $P_0(\mathbf{x}_0)$ at time t.

$$\begin{aligned}
&\frac{\partial n_1}{\partial t} - \frac{(n_2^2+n_3^2)\sqrt{n_1^2+n_3^2}}{n_3 g_{10}}\frac{\partial m}{\partial \xi_1} - \frac{n_1 n_2\sqrt{n_2^2+n_3^2}}{n_3 g_{20}}\frac{\partial m}{\partial \xi_2} = 0,\\
&\frac{\partial n_2}{\partial t} + \frac{n_1 n_2\sqrt{n_1^2+n_3^2}}{n_3 g_{10}}\frac{\partial m}{\partial \xi_1} + \frac{(n_1^2+n_3^2)\sqrt{n_2^2+n_3^2}}{n_3 g_{20}}\frac{\partial m}{\partial \xi_2} = 0,\\
&\frac{\partial m}{\partial t} - \frac{(m-1)\sqrt{n_1^2+n_3^2}}{2n_3 g_{10}}\frac{\partial m}{\partial \xi_1} + \frac{(m-1)\sqrt{n_2^2+n_3^2}}{2n_3 g_{20}}\frac{\partial n_2}{\partial \xi_2} = 0.
\end{aligned} \tag{55}$$

Eigenvalues of frozen equations

The eigenvalues of these three frozen equations from $|-\mu A + e_1 B^{(1)} + e_2 B^{(2)}| = 0$, with e_1, e_2 arbitrary, are

$$\mu_{1,2} = \pm\left[\frac{m-1}{2n_3^2}\left\{\left(n_2^2+n_3^2\right)\bar{e}_1^2 + 2n_1n_2\bar{e}_1\bar{e}_2 + \left(n_1^2+n_3^2\right)\bar{e}_2^2\right\}\right]^{1/2}, \tag{56}$$

$$\mu_3 = 0,$$

where

$$\bar{e}_1 = \frac{\sqrt{n_1^2+n_3^2}}{g_{10}}e_1, \quad \bar{e}_2 = \frac{\sqrt{n_2^2+n_3^2}}{g_{20}}e_2. \tag{57}$$

Here (e_1, e_2) is an arbitrary vector in $\mathbb{R}^2$.

Remark

The term in curly bracket in $\mu_{1,2}$ is positive definite. Therefore for $m > 1$, $\mu_1 = -\mu_2 > 0$ and hence the system (52) is strictly hyperbolic.

Eigenvalues and eigenvectors with frozen orthogonal vectors

Let $\mathbf{u}'$ and $\mathbf{v}'$ be frozen orthogonal vectors $P_0(\mathbf{x}_0)$, i.e,

$$\chi' = \frac{\pi}{2}, \ \cos\chi' = 1 \tag{58}$$

The eigenvalues $\nu_1, \nu_2, \ldots, \nu_7$ of (47) are given by

$$\det\begin{bmatrix} -\nu g_1' & 0 & \frac{mu_2'v_1'}{v_3'}e_1' & \frac{mu_1'v_1'}{v_3'}e_1' & -n_1e_1' & 0 & 0 \\ 0 & -\nu g_1' & \frac{mu_2'v_2'}{v_3'}e_1' & \frac{mu_1'v_2'}{v_3'} & -n_2e_1' & 0 & 0 \\ -\frac{mu_1'v_2'}{u_3'}e_2' & \frac{mu_1'v_1'}{u_3'}e_2' & -\nu g_2' & 0 & -n_1e_2' & 0 & 0 \\ -\frac{mu_2'v_2'}{u_3'}e_2' & \frac{mu_2'v_1'}{u_3'}e_2' & 0 & -\nu g_2' & -n_2e_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu\frac{2m}{m-1}g_1'g_2' & -\nu g_2' & -\nu g_1' \\ -\frac{mv_2'}{u_3'}e_1' & \frac{mv_1'}{u_3'}e_1' & 0 & 0 & 0 & -\nu & 0 \\ 0 & 0 & \frac{mu_2'}{v_3'}e_2' & -\frac{mu_1'}{v_3'}e_2' & 0 & 0 & -\nu \end{bmatrix} = 0. \tag{59}$$

Eigenvalues contd..

A long calculation leads to the following eigenvalues

$$\nu_{1,2} = \pm \left\{ \frac{(m-1)(e_1'^2 g_2' + e_2'^2 g_1')}{2 g_1'^2 g_2'^2} \right\}^{1/2}, \quad \nu_3 = \cdots = \nu_7 = 0. \tag{60}$$

Remark

1. *Number of independent eigenvectors for* $\nu_3 = \cdots = \nu_7 = 0$ *is only 4.*
2. $\Rightarrow$ *The system (52) is degenerate.*
3. *Eigenvalues are real for* $m > 1$ *and* ν_1 *and* ν_2 *are purely imaginary for* $m < 1$

Abstract of various forms of WNLRT

- Ray theory: 3 equations giving evolution of n_1, n_2 and m in WNLRT in terms of operators $\frac{\partial}{\partial t}, \frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}$ - a general case i.e., not just at a point in frozen coordinates.
- Ray theory: Above three equations in frozen orthogonal coordinates in terms of $\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}$ at a fixed point $P(\mathbf{x_0})$.
- WNLRT from KCL in differential form: The seven equations in frozen coordinates when $\mathbf{u}', \mathbf{v}'$ are orthogonal. The eigenvalues are explicitly calculated.
- WNLRT from KCL in differential form: The seven equations of WNLRT in most general coordinate system (ξ_1, ξ_2, t).

General case: Numerical computation of eigenvalues and eigenvectors.

- Failed to compute expressions for the eigenvalues and eigenvectors directly.
- The characteristic equation was solved numerically for a large number of values of parameters. Every case showed that
  1. $\lambda_3 = \lambda_4 = \cdots = \lambda_7 = 0$.
  2. $\lambda_1 = (-\lambda_2) \neq 0$ is real for $m > 1$ and purely imaginary for $m < 1$.
- We shall prove this by a very interesting method.

Transformation of coordinates

Transformation of coordinates (frozen at a point $P_0(\mathbf{x}_0)$) to get eigenvalues from orthogonal case to the general case.

- Orthogonal coordinates (η_1, η_2) with $\mathbf{u}'$ and $\mathbf{v}'$
- general coordinates (ξ_1, ξ_2) with $\mathbf{u}$ and $\mathbf{v}$.
- Suppose

$$\mathbf{u}' = \gamma_1 \mathbf{u} + \delta_1 \mathbf{v}, \;\; \mathbf{v}' = \gamma_2 \mathbf{u} + \delta_2 \mathbf{v}. \tag{61}$$

$\Rightarrow$ **Theorem:** Let λ' be an expression for an eigenvalue of equations in frozen orthogonal coordinates in terms of e_1'/g_1' and e_2'/g_2'. Then the expression for the same eigenvalue in the coordinates (ξ_1, ξ_2) in terms of e_1/g_1 and e_2/g_2 can be obtained from it by replacing e_1'/g_1' and e_2'/g_2' by

$$\lambda' = \lambda, \quad \frac{e_1'}{g_1'} = \gamma_1 \frac{e_1}{g_1} + \delta_1 \frac{e_2}{g_2}, \quad \frac{e_2'}{g_2'} = \gamma_2 \frac{e_1}{g_1} + \delta_2 \frac{e_2}{g_2}$$

Degeneracy in 3-D KCL

Thus we have proved the theorem

Theorem

The 3-D KCL system has 7 eigenvalues $\lambda_1, \lambda_2(= -\lambda_1), \lambda_3 = \lambda_4 = \cdots = \lambda_7 = 0$, where λ_1 and λ_2 are real for $m > 1$ and purely imaginary for $m < 1$. Further, the dimension of the eigenspace corresponding to the multiple eigenvalue 0 is 4.

Remark

1. *This important theorem completely describes the nature of the equations of WNLRT.*
2. *It shows that the system is degenerate and it would require considerable effort to solve it numerically even in the case $m > 1$.*
3. *We need to develop a numerical scheme which is genuinely multi-dimensional.*

An application of the transformation theory for determination of eigenvalues

- We know the eigenvalues when the coordinates η_1, η_2 on Ω_t are orthogonal.
- We also know the two operators $\frac{\partial}{\partial \xi_1}$ and $\frac{\partial}{\partial \xi_2}$ which appear in the three equations of the ray theory.
- We can find the linear relations between $\left(\frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}\right)$ and $\left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\right)$ and use the above theorem.
- This gives exactly the eigenvalues μ_1, μ_2 and μ_3 of the system (52).
- A wonderful result!

Conclusions

(0) The theory of 2-D KCL has no complication and many applications have been done.

(I) The problem of evolution of a moving surface in $\mathbb{R}^3$ is quite complex - its evolution is governed by

1. A system of ray equations along with a closure relation.
2. In ray theory formulation, it has a complete system of 3 equations - system (47)
3. This system is hyperbolic for $m > 1$ and has elliptic nature for $m < 1$.
4. This system can not describe kink type of singularities.

Conclusions contd..

(II) The KCL system (52) can describe all physically realistic features.

1. It is equivalent to a smaller system (47).
2. We have completely analyzed the KCL system - (52).
3. It has 7 eigenvalues, two non-zero and 5 are 0.
4. The system is degenerate but its nonzero eigenvalues are the same as those of the system (47), real for $m > 1$ and purely imaginary for $m < 1$.
5. For $m > 1$, the real eigenvalues carry with them changes in the geometry of Ω_t and the front velocity m. The zero eigenvalue transports the “energy” along a ray tube.

Formulation of ray coordinates

Let the initial wavefront be

$$\Omega_0 \colon x_3 = f(x_1, x_2). \tag{62}$$

On Ω_0 the ray coordinates (ξ_1, ξ_2) can be chosen to be

$$\xi_1 = x_1, \quad \xi_2 = x_2. \tag{63}$$

then

$$\Omega_0 \colon x_1 = \xi_1,\ x_2 = \xi_2,\ x_3 = f(\xi_1, \xi_2). \tag{64}$$

with this choice of (ξ_1, ξ_2), the initial values are

$$g_{1_0} = \sqrt{1 + f_{\xi_1}^2}, \quad g_{2_0} = \sqrt{1 + f_{\xi_2}^2}. \tag{65}$$

$$\mathbf{u}_0 = \frac{(1, 0, f_{\xi_1})}{\sqrt{1 + f_{\xi_1}^2}}, \quad \mathbf{v}_0 = \frac{(0, 1, f_{\xi_2})}{\sqrt{1 + f_{\xi_2}^2}}. \tag{66}$$

Let

$$m_0 = m_0(\xi_1, \xi_2) \tag{67}$$

be given.

Numerical Scheme

The system of conservation laws KCL+WNLRT transport equation for the variable $\mathbf{U} = (u_1, u_2, v_1, v_2, m, g_1, g_2)^T$ can be written as a system of conservation laws

$$(\mathbf{H}(\mathbf{U}))_t + (\mathbf{F_1}(\mathbf{U}))_{\xi_1} + (\mathbf{F_2}(\mathbf{U}))_{\xi_2} = 0, \tag{68}$$

where

$$\mathbf{H}(\mathbf{U}) = \left(g_1 u_1, g_1 u_2, g_1 u_3, g_2 v_1, g_2 v_2, g_2 v_3, (m-1)^2 e^{2(m-1)} g_1 g_2 \sin\chi\right)^T, \tag{69}$$

$$\mathbf{F_1}(\mathbf{U}) = (mn_1, mn_2, mn_3, 0, 0, 0, 0)^T, \tag{70}$$

$$\mathbf{F_2}(\mathbf{U}) = (0, 0, 0, mn_1, mn_2, mn_3, 0)^T. \tag{71}$$

Staggered central schemes for KCL

Note

In order to approximate the KCL system we apply both first order staggered Lax-Friedrichs scheme and second order Nessyahu-Tadmor scheme. Since central finite volume schemes are less dependent on hyperbolic structure of the underlying conservation laws they yield more robust solutions even for weakly hyperbolic problems (Enquist & Runborg).

An Example

The finite difference scheme is applied to a test case in which the initial wavefront Ω_0 has the shape of an elevated Gaussian pulse

$$\Omega_0 \colon x_3 = e^{-(x_1^2+x_2^2)} \equiv f(x_1, x_2), \tag{72}$$

On Ω_0 the ray coordinates ξ_1 and ξ_2 can be chosen to be

$$\xi_1 = x_1, \quad \xi_2 = x_2. \tag{73}$$

The initial values for the metrics g_1, g_2 and the vectors $\mathbf{u}$ and $\mathbf{v}$ can be obtained by

$$g_{10} = \left(1 + f_{\xi_1}^2\right)^{1/2}, \quad g_{20} = \left(1 + f_{\xi_2}^2\right)^{1/2}. \tag{74}$$

$$\mathbf{u}_0 = \frac{1}{g_1}\left(1, 0, f_{\xi_1}\right), \quad \mathbf{v}_0 = \frac{1}{g_2}\left(0, 1, f_{\xi_2}\right). \tag{75}$$

The initial value of m is set to $m_0 = 1.2$.

Evolution of Ω_t

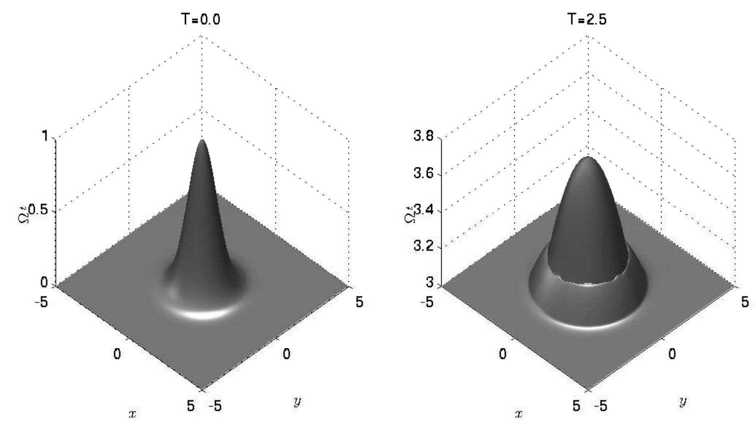


Figure: Nonlinear wavefront in the shape of a Gaussian pulse at time $T = 0$

Evolution of Ω_t

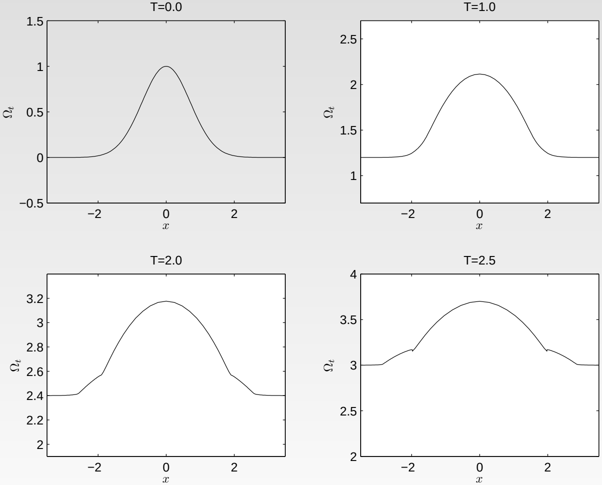


Figure: Wavefront at different times T

Evolution of the normal velocity m

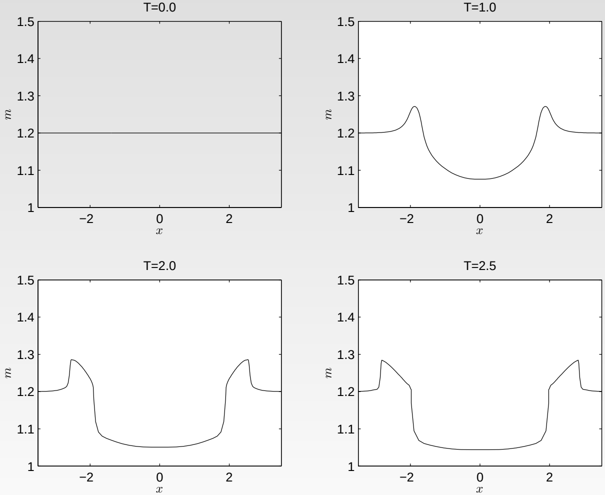


Figure: The variation of m at different times

Evolution of the metric g_1

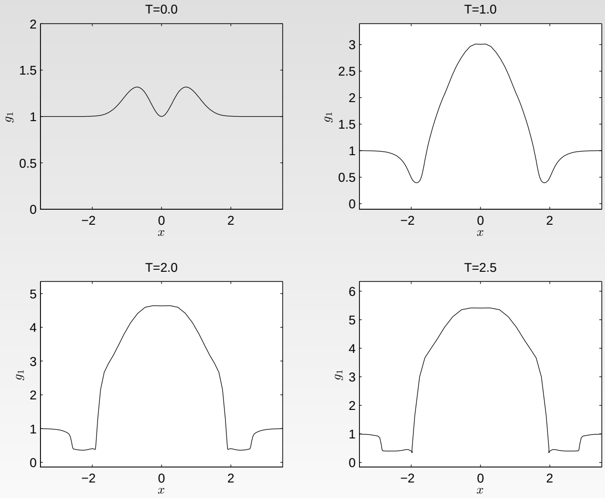


Figure: The evolution of the metric g_1 with time

Evolution of the metric g_2

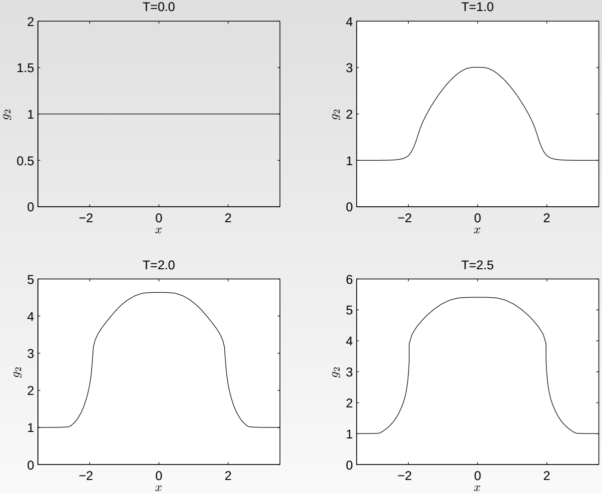


Figure: The variation of g_2 at different times

Second Example

The initial wavefront Ω_0 is in the shape of a paraboloid

$$\Omega_0 \colon x_3 = (x_1^2 + x_2^2) \equiv f(x_1, x_2), \tag{76}$$

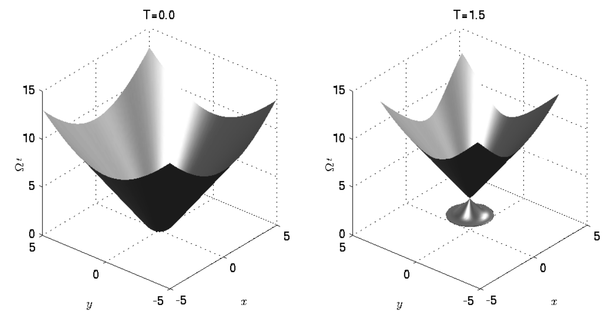


Figure: Nonlinear wavefront in the shape of a paraboloid at time $T = 0$

Thank You!