Does there exist an infinite $\sigma$-algebra which has only countable many members?

Solution: No. Suppose $\mathcal{M}$ be a $\sigma$-algebra on $X$ which has countably infinite members. For each $x \in X$ define $B_x = \cap_{x \in M \in \mathcal{M}} M$. Since $\mathcal{M}$ has countable members, so the intersection is over countable members or less, and so $B_x$ belongs $\mathcal{M}$, since $\mathcal{M}$ is closed under countable intersection. Define $\mathcal{N} = \{B_x \mid x \in X\}$. So $\mathcal{N} \subset \mathcal{M}$. Also we claim if $A, B \in \mathcal{N}$, with $A \neq B$, then $A \cap B = \emptyset$. Suppose $A \cap B \neq \emptyset$, then some $x \in A$ and same $x \in B$, but that would mean $A = B = \cap_{x \in M \in \mathcal{M}} M$. Hence $\mathcal{N}$ is a collection of disjoint subsets of $X$. Now if cardinality of $\mathcal{N}$ is finite say $n \in \mathbb{N}$, then it would imply cardinality of $\mathcal{M}$ is $2^n$, which is not the case. So cardinality of $\mathcal{N}$ should be at least $\aleph_0$. If cardinality of $\mathcal{N} = \aleph_0$, then cardinality of $\mathcal{M} = 2^{\aleph_0} = \aleph_1$, which is not possible as $\mathcal{M}$ has countable many members. Also if cardinality of $\mathcal{N} \geq \aleph_1$, so is the cardinality of $\mathcal{M}$, which again is not possible. So there does not exist an infinite $\sigma$-algebra having countable many members.
2 Prove an analogue of Theorem 1.8 for $n$ functions.

**Solution:** Analogous Theorem would be: Let $u_1, u_2, \ldots, u_n$ be real-valued measurable functions on a measurable space $X$, let $\Phi$ be a continuous map from $\mathbb{R}^n$ into topological space $Y$, and define

$$h(x) = \Phi(u_1(x), u_2(x), \ldots, u_n(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

**Proof:** Define $f : X \to \mathbb{R}^n$ such that $f(x) = (u_1(x), u_2(x), \ldots, u_n(x))$. So $h = \Phi \circ f$. So using Theorem 1.7, we only need to show $f$ is a measurable function. Consider a cube $Q$ in $\mathbb{R}^n$. $Q = I_1 \times I_2 \times \cdots \times I_n$, where $I_i$ are the intervals in $\mathbb{R}$. So

$$f^{-1}(Q) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap \cdots \cap u_n^{-1}(I_n)$$

Since each $u_i$ is measurable, so $f^{-1}(Q)$ is measurable for all cubes $Q \in \mathbb{R}^n$. But every open set $V$ in $\mathbb{R}^n$ is a countable union of such cubes, i.e $V = \bigcup_{i=1}^{\infty} Q_i$, therefore

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} Q_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(Q_i)$$

Since countable union of measurable sets is measurable, so $f^{-1}(V)$ is measurable. Hence $f$ is measurable.  \[ \square \]
3 Prove that if $f$ is a real function on a measurable space $X$ such that $\{x : f(x) \geq r\}$ is a measurable for every rational $r$, then $f$ is measurable.

**Solution:** Let $\mathcal{M}$ denote the $\sigma$-algebra of measurable sets in $X$. Let $\Omega$ be the collections of all $E \subset [-\infty, \infty]$ such that $f^{-1}(E) \in \mathcal{M}$. So for all rationals $r$, $[r, \infty] \in \Omega$. Let $\alpha \in \mathbb{R}$; we will show $(\alpha, \infty) \in \Omega$; hence from Theorem 1.12(c) conclude that $f$ is measurable.

Since rationals are dense in $\mathbb{R}$, therefore there exists a sequence of rationals $\{r_i\}$ such that $r_i > \alpha$ and $r_i \to \alpha$. Also $(\alpha, \infty) = \bigcup_{i=1}^{\infty} [r_i, \infty]$. Each $[r_i, \infty] \in \Omega$ and $\Omega$ is a $\sigma$-algebra (Theorem 1.12(a)) and hence closed under countable union; therefore $(\alpha, \infty) \in \Omega$. And so from Theorem 1.12(c), we conclude $f$ is measurable.
Let \( \{a_n\} \) and \( \{b_n\} \) be sequences in \([-\infty, \infty]\), prove the following assertions:

(a) \( \limsup_{x \to \infty} (-a_n) = -\liminf_{n \to \infty} a_n \). 

(b) \( \limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \)

provided none of the sums is of the form \( \infty - \infty \).

(c) If \( a_n \leq b_n \) for all \( n \), then

\( \limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n \).

Show by an example that the strict inequality can hold in (b).

**Solution:**

(a) We have for all \( n \in \mathbb{N} \),

\[
\sup_{i \geq n} \{-a_i\} = -\inf_{i \geq n} \{a_i\}
\]

taking limit \( n \to \infty \), we have desired equality.

(b) Again for all \( n \in \mathbb{N} \), we have

\[
\sup_{i \geq n} \{a_i + b_i\} \leq \sup_{i \geq n} \{a_i\} + \sup_{i \geq n} \{b_i\}
\]

Taking limit \( n \to \infty \), we have desired inequality.

(c) Since \( a_n \leq b_n \) for all \( n \), so for all \( n \) we have

\[
\inf_{i \geq n} \{a_i\} \leq \inf_{i \geq n} \{b_i\}
\]

Taking limit \( n \to \infty \), we have desired inequality.

For strict inequality in (b), consider \( a_n = (-1)^n \) and \( b_n = (-1)^{n+1} \).
5 (a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets 
\[ \{ x : f(x) < g(x) \}, \{ x : f(x) = g(x) \} \]
are measurable.

**Solution:** Given $f, g$ are measurable, therefore from 1.9(c) we conclude $g - f$ is also measurable. But then $\{ x : f(x) < g(x) \} = (g - f)^{-1}(0, \infty]$ is a measurable set by Theorem 1.12(c).

Also $\{ x : f(x) = g(x) \} = (g - f)^{-1}(0)$

\[ = (g - f)^{-1} \left( \bigcap \left( -\frac{1}{n}, \frac{1}{n} \right) \right) \]

\[ = \bigcap (g - f)^{-1} \left( -\frac{1}{n}, \frac{1}{n} \right) \]

Since each $(g - f)^{-1} \left( -\frac{1}{n}, \frac{1}{n} \right)$ is measurable, so is their countable intersection. Hence $\{ x : f(x) = g(x) \}$ is measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

**Solution:** Let $f_i$ be the sequence of real-measurable functions. Let $A$ denotes the set of points at which $f_i$ converges to a finite limit. But then

\[ A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{n} \bigcap_{i,j \geq m} \{ x : |f_i(x) - f_j(x)| < \frac{1}{n} \} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{n} (f_i - f_j)^{-1} \left( -\frac{1}{n}, \frac{1}{n} \right) \]

Since for each $i, j$, $f_i - f_j$ is measurable, so $(f_i - f_j)^{-1} \left( -\frac{1}{n}, \frac{1}{n} \right)$ is measurable too for all $n$. Also countable union and intersection of measurable sets is measurable, we conclude $A$ is measurable.
Let $X$ be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subset X$ such that either $E$ or $E^c$ is at most countable, and define $\mu(E) = 0$ in the first case and $\mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a $\sigma$-algebra in $X$ and that $\mu$ is a measure on $\mathcal{M}$. Describe the corresponding measurable functions and their integrals.

**Solution:** $\mathcal{M}$ is a $\sigma$-algebra in $X$: $X \in \mathcal{M}$, since $X^c = \emptyset$ is countable. Similarly $\emptyset \in \mathcal{M}$. Next if $A \in \mathcal{M}$, then either $A$ or $A^c$ is countable, that is either $(A^c)^c$ is countable or $A^c$ is countable; showing $A^c \in \mathcal{M}$. So $\mathcal{M}$ is closed under complement. Finally, we show $\mathcal{M}$ is closed under countable union. Suppose $A_i \in \mathcal{M}$ for $i \in \mathbb{N}$, we will show $\bigcup A_i$ also belongs to $\mathcal{M}$. If all $A_i$ are countable, so is their countable union, so $\bigcup A_i \in \mathcal{M}$. But when all $A_i$ are not countable means at least one say $A_j$ is uncountable. Then $A_j^c$ is countable. Also $(\bigcup A_i)^c \subset A_j^c$, showing $(\bigcup A_i)^c$ is countable. So $\bigcup A_i \in \mathcal{M}$. Hence $\mathcal{M}$ is closed under countable union.

$\mu$ is a measure on $\mathcal{M}$: Since $\mu$ takes values 0 and 1, therefore $\mu(A) \in [0, \infty]$ for all $A \in \mathcal{M}$. Next we show $\mu$ is countable additive. Let $A_i$ for $i \in \mathbb{N}$ are disjoint measurable sets. Define $A = \bigcup A_i$. We will show $\mu(A) = \sum \mu(A_i)$. If all $A_i$ are countable, so is $A$; therefore $\mu(A_i) = 0$ for all $i$ and $\mu(A) = 0$; and the equation $\mu(A) = \sum \mu(A_i)$ holds good. But when all $A_i$ are not countable means at least one say $A_j$ is uncountable. Since $A_j \in \mathcal{M}$, therefore $A_j^c$ is countable. Also Since all $A_i$ are disjoint, so for $i \neq j$, $A_i \in A_j^c$. So $\mu(A_i) = 0$ for $i \neq j$. Also $\mu(A_j) = \mu(A) = 1$ since both are uncountable. Hence $\mu(A) = \sum \mu(A_i)$.

Characterization of measurable functions and their integrals: Assume functions are real valued. First we isolate two class of measurable functions denoted by $F_\infty$ and $F_{-\infty}$, defines as:

$F_\infty = \{ f \mid f$ is measurable & $f^{-1}([\alpha, \infty])$ is countable for all $\alpha \in \mathbb{R} \}$

$F_{-\infty} = \{ f \mid f$ is measurable & $f^{-1}([\alpha, \infty])$ is countable for all $\alpha \in \mathbb{R} \}$

Next we characterize the reaming measurable functions. Since $f \notin F_\infty$ or $F_{-\infty}$, therefore $f^{-1}([\alpha, \infty])$ is uncountable for some $\alpha \in \mathbb{R}$. Therefore $\alpha_f$ defined as $\sup\{ \alpha \mid f^{-1}([\alpha, \infty])$ is countable $\}$ exists. So if $\beta > \alpha_f$, then $f^{-1}([\beta, \infty])$ is countable. Also if $\beta < \alpha_f$, then $f^{-1}([\beta, \infty]) = X - f^{-1}([\alpha_f, \infty])$. Since $f^{-1}([\beta, \infty])$ is uncountable and belongs to $\mathcal{M}$, therefore $X - f^{-1}([\beta, \infty])$ is countable. And so $f^{-1}(\alpha_f)$ is uncountable. Also
\( f^{-1}(\alpha_f) \in \mathcal{M} \), therefore \( f^{-1}(\gamma) \) is countable for all \( \gamma \neq \alpha_f \). Thus if \( f \) is a measurable function then either \( f \in F_{\infty} \) or \( F_{-\infty} \), or there exists \( \alpha_f \in \mathbb{R} \) such that \( f^{-1}(\alpha_f) \) is uncountable while \( f^{-1}(\beta) \) is countable for all \( \beta \neq \alpha_f \). Once we have characterization, integrals are easy to describe:

\[
\int_X f \, d\mu = \begin{cases} 
\infty & \text{if } f \in F_{\infty} \\
-\infty & \text{if } f \in F_{-\infty} \\
\alpha_f & \text{else}
\end{cases}
\]
Suppose \( f_n : X \to [0, \infty] \) is measurable for \( n = 1, 2, 3, \ldots, f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0 \), \( f_n(x) \to f(x) \) as \( n \to \infty \), for every \( x \in X \), and \( f_1 \in L^1(\mu) \). Prove that then
\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu
\]
and show that this conclusion does not follow if the condition “\( f_1 \in L^1(\mu) \)” is omitted.

**Solution:** Take \( g = f_1 \) in the Theorem 1.34 to conclude
\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu
\]
For showing \( f_1 \in L^1(\mu) \) is a necessary condition for the conclusion, take \( X = \mathbb{R} \) and \( f_n = \chi_{[n, \infty)} \). So we have \( f(x) = 0 \) for all \( x \), and therefore \( \int_X f \, d\mu = 0 \). While \( \int_X f_n \, d\mu = \infty \) for all \( n \).
8 Put $f_n = \chi_E$ if $n$ is odd, $f_n = 1 - \chi_E$ if $n$ is even. What is the relevance of this example to Fatou’s lemma?

Solution: With the described sequence of $f_n$, strict inequality occurs in Fatou’s Lemma (1.28). We have

$$\int_X \left( \liminf_{n \to \infty} f_n \right) \, d\mu = 0$$

While

$$\liminf_{n \to \infty} \int_X f_n \, d\mu = \min(\mu(E), \mu(X) - \mu(E)) \neq 0,$$

assuming $\mu(X) \neq \mu(E)$. \qed
Suppose $\mu$ is a positive measure on $X$, $f : X \to [0, \infty]$ is measurable, $\int_X f \, d\mu = c$, where $0 < c < \infty$, and $\alpha$ is a constant. Prove that

$$\lim_{n \to \infty} \int_X n \log[1 + (f/n)^\alpha] \, d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, prove that the integrands is dominated are dominated by $\alpha f$. If $\alpha < 1$, Fatou’s lemma can be applied.

Solution: As given in the hint, we consider two cases for $\alpha$:

Case when $0 < \alpha < 1$: Define $\phi_n(x) = n \log(1 + (f(x)/n)^\alpha)$. Since $\phi_n : X \to [0, \infty]$, therefore Fatou’s lemma is applicable. So

$$\int_X (\liminf_{n \to \infty} \phi_n) \, d\mu \leq \liminf_{n \to \infty} \int_X \phi_n \, d\mu$$

Also

$$\lim_{n \to \infty} n \log(1 + (f(x)/n)^\alpha) = \lim_{n \to \infty} \frac{1}{1 + (f(x)/n)^\alpha} \frac{-\alpha f(x)^\alpha}{n^{\alpha+1}} = \frac{\alpha n^{1-\alpha} f(x)^\alpha}{1 + (f(x)/n)^\alpha}$$

Since $\alpha < 1$ and $\int_X f \, d\mu < \infty$, therefore

$$\lim_{n \to \infty} n \log(1 + (f(x)/n)^\alpha) = \infty \text{ a.e } x \in X$$

And hence $\int_X (\liminf_{n \to \infty} \phi_n) \, d\mu = \infty$. Therefore

$$\lim_{n \to \infty} \int_X \phi_n \, d\mu \geq \liminf_{n \to \infty} \int_X \phi_n \, d\mu = \infty$$

Case when $\alpha \geq 1$: We claim $\phi_n(x)$ is dominated by $\alpha f(x)$. For a.e. $x \in X$ and $\alpha \geq 1$, we need to show

$$n \log(1 + (f(x)/n)^\alpha) \leq \alpha f(x) \text{ for all } n$$

i.e. $\log \left( 1 + \left( \frac{f(x)}{n} \right)^\alpha \right) \leq \alpha \frac{f(x)}{n}$

(1)

Define $g(\lambda) = \log(1 + \lambda^\alpha) - \alpha \lambda$ for $\lambda \geq 0$. So if $g(\lambda) \leq 0$ for $\alpha \geq 1$ and $\lambda \geq 0$, then (1) follows by taking $\lambda = f(x)/n$. So we need show $g(\lambda) \leq 0$ for $\alpha \geq 1$ and $\lambda \geq 0$. Computing derivative of $g$, we have

$$g'(\lambda) = -\frac{\alpha(1 + \lambda^\alpha - \lambda^{\alpha-1})}{1 + \lambda^\alpha}$$
When $0 \geq \lambda \geq 1$, we have $1 - \lambda^{\alpha - 1} \geq 0$; while when $\lambda > 1$, we have $\lambda^\alpha - \lambda^{\alpha - 1} > 0$. Thus $g'(\lambda) \leq 0$. Also $g(0) = 0$, therefore $g(\lambda) \leq 0$ for all $\lambda \geq 0$ and $\alpha \geq 1$. And so for $\alpha \geq 1$ we have $\log(1 + (f(x)/n)^\alpha) \leq \alpha f(x)$ for all $n$ and a.e $x \in X$. Since $\alpha f(x) \in L^1(\mu)$, DCT(Theorem 1.34) is applicable. Thus

$$
\lim_{n \to \infty} \int_X n \log(1 + (f/n)^\alpha) \, d\mu = \int_X n \log(1 + (f/n)^\alpha) \, d\mu
$$

When $\alpha = 1$, $\lim_{n \to \infty} (n \log(1 + (f/n)^\alpha)) = f(x)$ (calculating the same as calculated for the case $\alpha < 1$). And when $\alpha > 1$, we have $\lim_{n \to \infty} (n \log(1 + (f/n)^\alpha)) = 0$. And hence

$$
\lim_{n \to \infty} \int_X n \log(1 + (f/n)^\alpha) \, d\mu = \begin{cases} 
\infty & \text{if } 0 < \alpha < 1, \\
\ c & \text{if } \alpha = 1, \\
\ 0 & \text{if } 1 < \alpha < \infty.
\end{cases}
$$
10 Suppose \( \mu(X) < \infty \), \( \{ f_n \} \) is a sequence of bounded complex measurable functions on \( X \), and \( f_n \to f \) uniformly on \( X \). Prove that

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu,
\]

and show that the hypothesis “\( \mu(X) < \infty \)” cannot be omitted.

**Solution:** Let \( \epsilon > 0 \). Since \( f_n \to f \) uniformly, therefore there exists \( n_0 \in \mathbb{N} \) such that

\[
|f_n(x) - f(x)| < \epsilon \quad \forall \, n \geq n_0
\]

Therefore \( |f(x)| < |f_{n_0}(x)| + \epsilon \). Also \( |f_n(x)| < |f(x)| + \epsilon \). Combining both equations, we get

\[
|f_n(x)| < |f_{n_0}| + 2\epsilon \quad \forall \, n \geq n_0
\]

Define \( g(x) = \max(|f_1(x)|, \ldots, |f_{n_0-1}(x)|, |f_{n_0}(x)| + 2\epsilon) \), then \( f_n(x) \leq g(x) \) for all \( n \). Also \( g \) is bounded. Since \( \mu(X) < \infty \), therefore \( g \in L^1(\mu) \). Now apply DCT(Theorem 1.34) to get

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu
\]

To show “\( \mu(X) < \infty \)” is a necessary condition, consider \( X = \mathbb{R} \) with usual measure \( \mu \), and \( f_n(x) = \frac{1}{n} \). We have \( \lim_{n \to \infty} \int_X f_n \, d\mu = \infty \), while \( \int_X f \, d\mu = 0 \), since \( f = 0 \).

**REMARK:** The condition “\( f_n \to f \) uniformly” is also a necessary condition.
11 Show that

\[ A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \]

in Theorem 1.41, and hence prove the theorem without any reference to integration.

Solution: \( A \) is defined as the collections of all \( x \) which lie in infinitely many \( E_k \). Thus \( x \in A \iff x \in \bigcup_{k=n}^{\infty} E_k \quad \forall n \in \mathbb{N} \); and so

\[ A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \]

Now let \( \epsilon > 0 \). Since \( \sum_{k=1}^{\infty} \mu(E_k) < \infty \), therefore there exists \( n_0 \in \mathbb{N} \) such that \( \sum_{k=n_0}^{\infty} \mu(E_k) < \epsilon \). And

\[
\mu(A) = \mu\left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right) \\
\leq \mu\left( \bigcup_{k=n_0}^{\infty} E_k \right) \\
\leq \sum_{k=n_0}^{\infty} \mu(E_k) \\
< \epsilon
\]

Make \( \epsilon \to 0 \) to conclude \( \mu(A) = 0 \).
Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

**Solution:** Let $(X, \mathcal{M}, \mu)$ be the measure space. Suppose the statement is not true. Therefore there exists a $\epsilon > 0$ such that there exists no $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$. That means for each $\delta > 0$, there exists a $E_\delta \in \mathcal{M}$ such that $\mu(E_\delta) < \delta$, while $\int_{E_\delta} |f| \, d\mu > \epsilon$. By taking $\delta = 1/2^n$, where $n \in \mathbb{N}$, we construct a sequence of measurable sets $\{E_{1/2^n}\}$, such that $\mu(E_{1/2^n}) < 1/2^n$ for all $n$ and $\int_{E_{1/2^n}} |f| \, d\mu > \epsilon$.

Now define $A_k = \bigcup_{n=k}^\infty E_{1/2^n}$ and $A = \bigcap_{k=1}^\infty A_k$. We have $A_1 \supset A_2 \supset A_3 \cdots$, and $\mu(A_1) = \mu(\bigcup_{n=1}^\infty E_{1/2^n}) \leq \sum_{n=1}^\infty \mu(E_{1/2^n}) < \sum_{n=1}^\infty 1/2^n < \infty$. Therefore from Theorem 1.19(e), we conclude $\mu(A_k) \to \mu(A)$.

Next define $\phi : \mathcal{M} \to [0, \infty]$ such that $\phi(E) = \int_E |f| \, d\mu$. Clearly, by Theorem 1.29, $\phi$ is a measure on $\mathcal{M}$. Therefore, again by Theorem 1.19(e), we have $\phi(A_k) \to \phi(A)$. Since $A = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_{1/2^n}$, therefore from previous Exercise, we get $\mu(A) = 0$. Therefore $\phi(A) = \int_A |f| \, d\mu = 0$. While

$$\phi(A_k) = \phi\left(\bigcup_{n=k}^\infty E_{1/2^n}\right) \geq \phi(E_{1/2^k}) = \int_{E_{1/2^k}} |f| \, d\mu > \epsilon$$

Therefore $\phi(A_k) \not\to \phi(A)$, a contradiction. Hence the result.  

12
Show that proposition 1.24(c) is also true when $c = \infty$.

**Solution:** We have to show

$$\int_X cf \, d\mu = c \int_X f \, d\mu, \text{ when } c = \infty \text{ and } f \geq 0$$

We consider two cases: when $\int_X f \, d\mu = 0$ and $\int_X f \, d\mu > 0$. When $\int_X f \, d\mu = 0$, we have from Theorem 1.39(a), $f = 0$ a.e., therefore, $cf = 0$ a.e.. And hence

$$\int_X cf \, d\mu = 0 = c \int_X f \, d\mu$$

While when $\int_X f \, d\mu > 0$, there exist a $\epsilon > 0$ and a measurable set $E$, such that $\mu(E) > 0$ and $f(x) > \epsilon$ whenever $x \in E$; because otherwise $f(x) < \epsilon$ a.e. for all $\epsilon > 0$; making $\epsilon \to 0$, we get $f(x) = 0$ a.e. and hence $\int_X f \, d\mu = 0$, which is not the case. But then

$$\int_X cf \, d\mu \geq \int_E cf \, d\mu > \epsilon \int_E c \, d\mu = \infty$$

Also $c \int_X f \, d\mu = \infty$

Hence the proposition is true for $c = \infty$ too.  

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16